4 □ APPLICATIONS OF DIFFERENTIATION

4.1 Maximum and Minimum Values

1. A function f has an **absolute minimum** at $x = c$ if $f(c)$ is the smallest function value on the entire domain of f, whereas f has a **local minimum** at c if $f(c)$ is the smallest function value when x is near c.

2. (a) The Extreme Value Theorem

(b) See the Closed Interval Method.

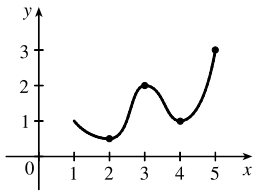
3. Absolute maximum at s, absolute minimum at r, local maximum at c, local minima at b and r, neither a maximum nor a minimum at a and d.

4. Absolute maximum at r; absolute minimum at a; local maxima at b and r; local minimum at d; neither a maximum nor a minimum at c and s.

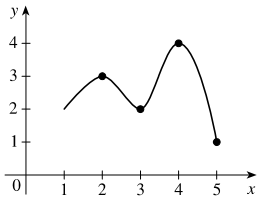
5. Absolute maximum value is $f(4) = 5$; there is no absolute minimum value; local maximum values are $f(4) = 5$ and $f(6) = 4$; local minimum values are $f(2) = 2$ and $f(1) = f(5) = 3$.

6. There is no absolute maximum value; absolute minimum value is $g(4) = 1$; local maximum values are $g(3) = 4$ and $g(6) = 3$; local minimum values are $g(2) = 2$ and $g(4) = 1$.

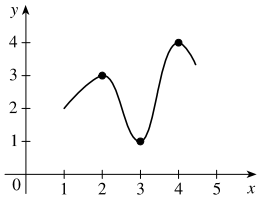
7. Absolute maximum at 5, absolute minimum at 2, local maximum at 3, local minima at 2 and 4



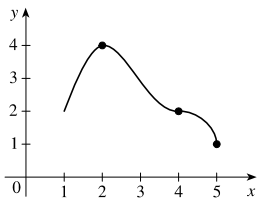
8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3



9. Absolute minimum at 3, absolute maximum at 4, local maximum at 2



10. Absolute maximum at 2, absolute minimum at 5, 4 is a critial number but there is no local maximum or minimum there.



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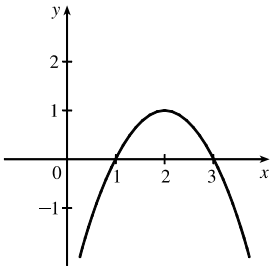
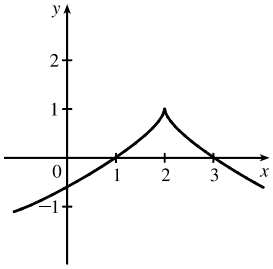
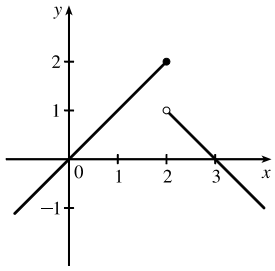
11. (a)



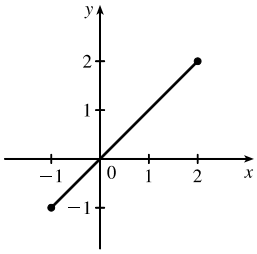
(b)



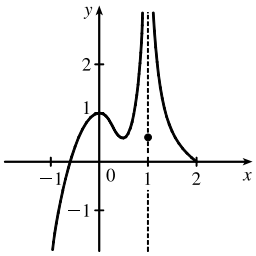
(c)

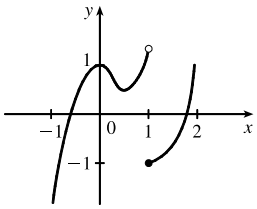


12. (a) Note that a local maximum cannot occur at an endpoint.



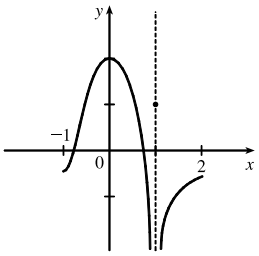
(b)



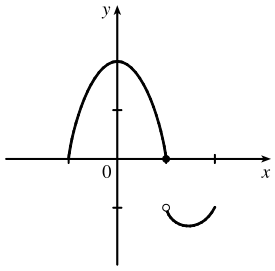


Note: By the Extreme Value Theorem, f must *not* be continuous.

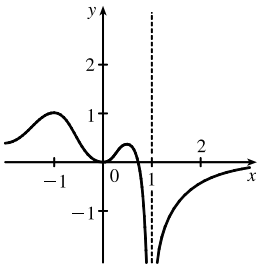
13. (a) *Note:* By the Extreme Value Theorem, f must *not* be continuous; because if it were, it would attain an absolute minimum.



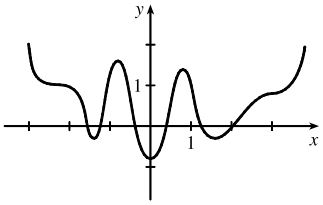
(b)



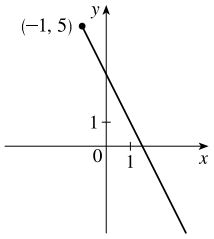
14. (a)



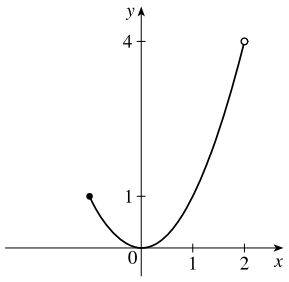
(b)



15. $f(x) = 3 - 2x,\ x \geq -1$. Absolute maximum $f(-1) = 5$; no local maximum. No absolute or local minimum.



16. $f(x) = x^2,\ -1 \leq x < 2$. No absolute or local maximum. Absolute and local minimum $f(0) = 0$.



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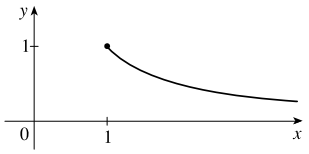
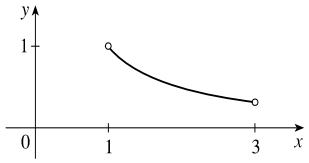
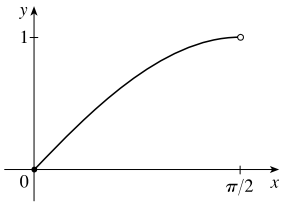
17. $f(x) = 1/x,\ x \geq 1$. Absolute maximum $f(1) = 1$; no local maximum. No absolute or local minimum.



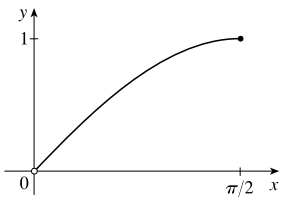
18. $f(x) = 1/x,\ 1 < x < 3$. No absolute or local maximum. No absolute or local minimum.



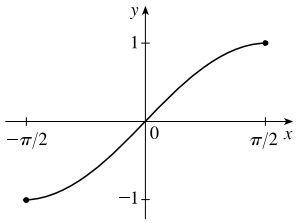
19. $f(x) = \sin x,\ 0 \leq x < \pi/2$. No absolute or local maximum. Absolute minimum $f(0) = 0$; no local minimum.



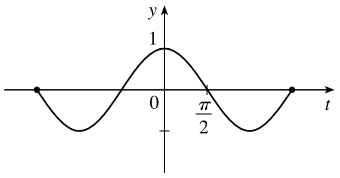
20. $f(x) = \sin x,\ 0 < x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. No absolute or local minimum.



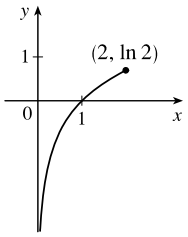
21. $f(x) = \sin x,\ -\pi/2 \leq x \leq \pi/2$. Absolute maximum $f\left(\frac{\pi}{2}\right) = 1$; no local maximum. Absolute minimum $f\left(-\frac{\pi}{2}\right) = -1$; no local minimum.



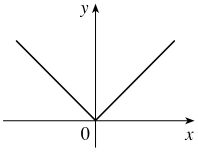
22. $f(t) = \cos t,\ -\frac{3\pi}{2} \leq t \leq \frac{3\pi}{2}$. Absolute and local maximum $f(0) = 1$; absolute and local minima $f(\pm\pi, -1)$.



23. $f(x) = \ln x,\ 0 < x \leq 2$. Absolute maximum $f(2) = \ln 2 \approx 0.69$; no local maximum. No absolute or local minimum.



24. $f(x) = |\ x\ |$. No absolute or local maximum. Absolute and local minimum $f(0) = 0$.



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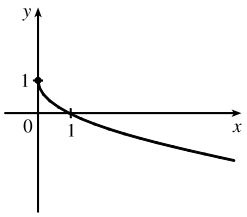
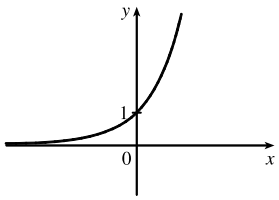
25. $f(x) = 1 - \sqrt{x}$. Absolute maximum $f(0) = 1$; no local maximum. No absolute or local minimum.

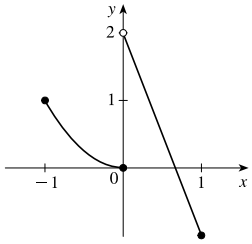


26. $f(x) = e^x$. No absolute or local maximum or minimum value.



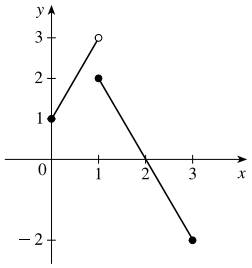
27. $f(x) = \begin{cases} x^2 & \text{if } -1 \le x \le 0 \\ 2 - 3x & \text{if } 0 < x \le 1 \end{cases}$

No absolute or local maximum.
Absolute minimum $f(1) = -1$.
Local minimum $f(0) = 0$.



28. $f(x) = \begin{cases} 2x + 1 & \text{if } 0 \le x < 1 \\ 4 - 2x & \text{if } 1 \le x \le 3 \end{cases}$

No absolute or local maximum.
Absolute minimum $f(3) = -2$; no local minimum.



29. $f(x) = 3x^2 + x - 2 \;\Rightarrow\; f'(x) = 6x + 1$. $f'(x) = 0 \;\Rightarrow\; x = -\frac{1}{6}$. This is the only critical number.

30. $g(v) = v^3 - 12v + 4 \;\Rightarrow\; g'(v) = 3v^2 - 12 = 3(v^2 - 4) = 3(v + 2)(v - 2)$. $g'(v) = 0 \;\Rightarrow\; v = -2, 2$. These are the only critical numbers.

31. $f(x) = 3x^4 + 8x^3 - 48x^2 \;\Rightarrow\; f'(x) = 12x^3 + 24x^2 - 96x = 12x(x^2 + 2x - 8) = 12x(x + 4)(x - 2)$.
$f'(x) = 0 \;\Rightarrow\; x = -4, 0, 2$. These are the only critical numbers.

32. $f(x) = 2x^3 + x^2 + 8x \;\Rightarrow\; f'(x) = 6x^2 + 2x + 8 = 2(3x^2 + x + 4)$. Using the quadratic formula, $f'(x) = 0 \;\Leftrightarrow\; x = \dfrac{-1 \pm \sqrt{-47}}{6}$. Since the discriminant, -47, is negative, there are no real solutions, and hence, there are no critical numbers.

33. $g(t) = t^5 + 5t^3 + 50t \;\Rightarrow\; g'(t) = 5t^4 + 15t^2 + 50 = 5(t^4 + 3t^2 + 10)$. Using the quadratic formula to solve for t^2, $g'(t) = 0 \;\Leftrightarrow\; t^2 = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(10)}}{2(1)} = \dfrac{-3 \pm \sqrt{-31}}{2}$. Since the discriminant, -31, is negative, there are no real solutions, and hence, there are no critical numbers.

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34. $A(x) = |3-2x| = \begin{cases} 3-2x & \text{if } 3-2x \ge 0 \\ -(3-2x) & \text{if } 3-2x < 0 \end{cases} = \begin{cases} 3-2x & \text{if } x \le \frac{3}{2} \\ 2x-3 & \text{if } x > \frac{3}{2} \end{cases}$

$A'(x) = \begin{cases} -2 & \text{if } x < \frac{3}{2} \\ 2 & \text{if } x > \frac{3}{2} \end{cases}$ and $A'(x)$ does not exist at $x = \dfrac{3}{2}$, so $x = \dfrac{3}{2}$ is a critical number.

35. $g(y) = \dfrac{y-1}{y^2-y+1} \quad \Rightarrow$

$$g'(y) = \frac{(y^2-y+1)(1)-(y-1)(2y-1)}{(y^2-y+1)^2} = \frac{y^2-y+1-(2y^2-3y+1)}{(y^2-y+1)^2} = \frac{-y^2+2y}{(y^2-y+1)^2} = \frac{y(2-y)}{(y^2-y+1)^2}.$$

$g'(y) = 0 \quad \Rightarrow \quad y = 0, 2$. The expression $y^2 - y + 1$ is never equal to 0, so $g'(y)$ exists for all real numbers.

The critical numbers are 0 and 2.

36. $h(p) = \dfrac{p-1}{p^2+4} \quad \Rightarrow \quad h'(p) = \dfrac{(p^2+4)(1)-(p-1)(2p)}{(p^2+4)^2} = \dfrac{p^2+4-2p^2+2p}{(p^2+4)^2} = \dfrac{-p^2+2p+4}{(p^2+4)^2}.$

$h'(p) = 0 \quad \Rightarrow \quad p = \dfrac{-2 \pm \sqrt{4+16}}{-2} = 1 \pm \sqrt{5}$. The critical numbers are $1 \pm \sqrt{5}$. [$h'(p)$ exists for all real numbers.]

37. $p(x) = \dfrac{x^2+2}{2x-1} \quad \Rightarrow$

$$p'(x) = \frac{(2x-1)(2x)-(x^2+2)(2)}{(2x-1)^2} = \frac{4x^2-2x-2x^2-4}{(2x-1)^2} = \frac{2x^2-2x-4}{(2x-1)^2} = \frac{2(x^2-x-2)}{(2x-1)^2} = \frac{2(x-2)(x+1)}{(2x-1)^2}.$$

$p'(x) = 0 \quad \Rightarrow \quad x = -1$ or 2. $p'(x)$ does not exist at $x = \frac{1}{2}$, but $\frac{1}{2}$ is not in the domain of p, so the critical numbers are -1 and 2.

38. $q(t) = \dfrac{t^2+9}{t^2-9} \quad \Rightarrow \quad q'(t) = \dfrac{(t^2-9)(2t)-(t^2+9)(2t)}{(t^2-9)^2} = \dfrac{2t^3-18t-2t^3-18t}{(t^2-9)^2} = -\dfrac{36t}{(t^2-9)^2}$. $q'(t) = 0 \quad \Rightarrow$

$36t = 0 \quad \Rightarrow \quad t = 0$. $q'(t)$ does not exist at $t = \pm 3$, but 3 and -3 are not in the domain of q, so 0 is the only critical number.

39. $h(t) = t^{3/4} - 2t^{1/4} \quad \Rightarrow \quad h'(t) = \frac{3}{4}t^{-1/4} - \frac{2}{4}t^{-3/4} = \frac{1}{4}t^{-3/4}(3t^{1/2}-2) = \dfrac{3\sqrt{t}-2}{4\sqrt[4]{t^3}}.$

$h'(t) = 0 \quad \Rightarrow \quad 3\sqrt{t} = 2 \quad \Rightarrow \quad \sqrt{t} = \frac{2}{3} \quad \Rightarrow \quad t = \frac{4}{9}$. $h'(t)$ does not exist at $t = 0$, so the critical numbers are 0 and $\frac{4}{9}$.

40. $g(x) = \sqrt[3]{4-x^2} = (4-x^2)^{1/3} \quad \Rightarrow \quad g'(x) = \frac{1}{3}(4-x^2)^{-2/3}(-2x) = \dfrac{-2x}{3(4-x^2)^{2/3}}$. $g'(x) = 0 \quad \Rightarrow \quad x = 0$.

$g'(\pm 2)$ do not exist. Thus, the three critical numbers are -2, 0, and 2.

41. $F(x) = x^{4/5}(x-4)^2 \quad \Rightarrow$

$$F'(x) = x^{4/5} \cdot 2(x-4) + (x-4)^2 \cdot \tfrac{4}{5}x^{-1/5} = \tfrac{1}{5}x^{-1/5}(x-4)[5 \cdot x \cdot 2 + (x-4) \cdot 4]$$
$$= \frac{(x-4)(14x-16)}{5x^{1/5}} = \frac{2(x-4)(7x-8)}{5x^{1/5}}$$

$F'(x) = 0 \quad \Rightarrow \quad x = 4, \frac{8}{7}$. $F'(0)$ does not exist. Thus, the three critical numbers are 0, $\frac{8}{7}$, and 4.

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42. $h(x) = x^{-1/3}(x-2) \Rightarrow$

$h'(x) = x^{-1/3}\cdot 1 + (x-2)\cdot\left(-\frac{1}{3}x^{-4/3}\right) = \frac{1}{3}x^{-4/3}\,[3x + (x-2)(-1)] = \dfrac{2x+2}{3x^{4/3}} = \dfrac{2(x+1)}{3x^{4/3}}$. $h'(x) = 0 \Rightarrow$

$2(x+1) = 0 \Rightarrow x = -1$. $h'(x)$ does not exist at $x = 0$, but 0 is not in the domain of h, so -1 is the only critical number.

43. $f(x) = x^{1/3}(4-x)^{2/3} \Rightarrow$

$f'(x) = x^{1/3}\cdot\frac{2}{3}(4-x)^{-1/3}\cdot(-1) + (4-x)^{2/3}\cdot\frac{1}{3}x^{-2/3} = \frac{1}{3}x^{-2/3}(4-x)^{-1/3}[-2x + (4-x)] = \dfrac{4-3x}{3x^{2/3}(4-x)^{1/3}}$.

$f'(x) = 0 \Rightarrow 4 - 3x = 0 \Rightarrow x = \frac{4}{3}$. $f'(0)$ and $f'(4)$ are undefined. Thus, the three critical numbers are 0, $\frac{4}{3}$, and 4.

44. $f(\theta) = \theta + \sqrt{2}\cos\theta \Rightarrow f'(\theta) = 1 - \sqrt{2}\sin\theta$. $f'(0) = 0 \Rightarrow 1 - \sqrt{2}\sin\theta = 0 \Rightarrow \sin\theta = \dfrac{1}{\sqrt{2}} \Rightarrow$

$\theta = \frac{\pi}{4} + 2n\pi$ [n an integer], $\theta = \frac{3\pi}{4} + 2n\pi$ are critical numbers. [$f'(\theta)$ exists for all real numbers.]

45. $f(\theta) = 2\cos\theta + \sin^2\theta \Rightarrow f'(\theta) = -2\sin\theta + 2\sin\theta\,\cos\theta$. $f'(\theta) = 0 \Rightarrow 2\sin\theta\,(\cos\theta - 1) = 0 \Rightarrow \sin\theta = 0$ or $\cos\theta = 1 \Rightarrow \theta = n\pi$ [n an integer] or $\theta = 2n\pi$. The solutions $\theta = n\pi$ include the solutions $\theta = 2n\pi$, so the critical numbers are $\theta = n\pi$.

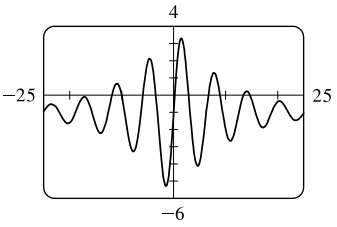
46. $p(t) = te^{4t} \Rightarrow p'(t) = t(4e^{4t}) + e^{4t}(1) = e^{4t}(4t+1)$. $p'(t) = 0 \Rightarrow t = -\frac{1}{4}$ [e^{4t} is never equal to 0]. $p'(t)$ always exists, so the only critical number is $-\frac{1}{4}$.

47. $g(x) = x^2\ln x \Rightarrow g'(x) = x^2(1/x) + (\ln x)(2x) = x + 2x\ln x = x(1 + 2\ln x)$. $g'(x) = 0 \Rightarrow 1 + 2\ln x = 0$ [or $x = 0$, but 0 is not in the domain of g] $\Rightarrow \ln x = -\frac{1}{2} \Rightarrow x = e^{-1/2}$. $g'(0)$ does not exist, but 0 is not in the domain of g, so the only critical number is $e^{-1/2} = 1/\sqrt{e}$.

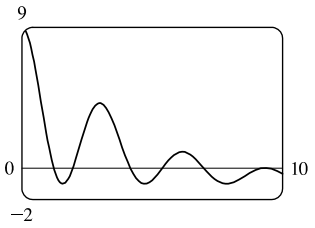
48. $B(u) = 4\tan^{-1}u - u \Rightarrow B'(u) = \dfrac{4}{1+u^2} - 1 = \dfrac{4}{1+u^2} - \dfrac{1+u^2}{1+u^2} = \dfrac{3-u^2}{1+u^2}$. $B'(u) = 0 \Rightarrow 3 - u^2 = 0 \Rightarrow$

$u = -\sqrt{3}, \sqrt{3}$. The expression $1 + u^2$ is never equal to 0, so $B'(u)$ exists for all real numbers. Thus, the critical numbers are $-\sqrt{3}$ and $\sqrt{3}$.

49. The graph of $f'(x) = 5e^{-0.1|x|}\sin x - 1$ has 10 zeros and exists everywhere, so f has 10 critical numbers.



50. A graph of $f'(x) = \dfrac{100\cos^2 x}{10 + x^2} - 1$ is shown. There are 7 zeros between 0 and 10, and 7 more zeros since f' is an even function. f' exists everywhere, so f has 14 critical numbers.



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51. $f(x) = 12 + 4x - x^2,\ [0, 5]$. $f'(x) = 4 - 2x = 0 \quad \Leftrightarrow \quad x = 2$. $f(0) = 12$, $f(2) = 16$, and $f(5) = 7$. So $f(2) = 16$ is the absolute maximum value and $f(5) = 7$ is the absolute minimum value.

52. $f(x) = 5 + 54x - 2x^3,\ [0, 4]$. $f'(x) = 54 - 6x^2 = 6(9 - x^2) = 6(3 + x)(3 - x) = 0 \quad \Leftrightarrow \quad x = -3, 3$. $f(0) = 5$, $f(3) = 113$, and $f(4) = 93$. So $f(3) = 113$ is the absolute maximum value and $f(0) = 5$ is the absolute minimum value.

53. $f(x) = 2x^3 - 3x^2 - 12x + 1,\ [-2, 3]$. $f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2) = 6(x - 2)(x + 1) = 0 \quad \Leftrightarrow$ $x = 2, -1$. $f(-2) = -3$, $f(-1) = 8$, $f(2) = -19$, and $f(3) = -8$. So $f(-1) = 8$ is the absolute maximum value and $f(2) = -19$ is the absolute minimum value.

54. $f(x) = x^3 - 6x^2 + 5,\ [-3, 5]$. $f'(x) = 3x^2 - 12x = 3x(x - 4) = 0 \quad \Leftrightarrow \quad x = 0, 4$. $f(-3) = -76$, $f(0) = 5$, $f(4) = -27$, and $f(5) = -20$. So $f(0) = 5$ is the absolute maximum value and $f(-3) = -76$ is the absolute minimum value.

55. $f(x) = 3x^4 - 4x^3 - 12x^2 + 1,\ [-2, 3]$. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x + 1)(x - 2) = 0 \quad \Leftrightarrow$ $x = -1, 0, 2$. $f(-2) = 33$, $f(-1) = -4$, $f(0) = 1$, $f(2) = -31$, and $f(3) = 28$. So $f(-2) = 33$ is the absolute maximum value and $f(2) = -31$ is the absolute minimum value.

56. $f(t) = (t^2 - 4)^3,\ [-2, 3]$. $f'(t) = 3(t^2 - 4)^2(2t) = 6t(t + 2)^2(t - 2)^2 = 0 \quad \Leftrightarrow \quad t = -2, 0, 2$. $f(\pm 2) = 0$, $f(0) = -64$, and $f(3) = 5^3 = 125$. So $f(3) = 125$ is the absolute maximum value and $f(0) = -64$ is the absolute minimum value.

57. $f(x) = x + \dfrac{1}{x},\ [0.2, 4]$. $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x + 1)(x - 1)}{x^2} = 0 \quad \Leftrightarrow \quad x = \pm 1$, but $x = -1$ is not in the given interval, $[0.2, 4]$. $f'(x)$ does not exist when $x = 0$, but 0 is not in the given interval, so 1 is the only critical nuumber. $f(0.2) = 5.2$, $f(1) = 2$, and $f(4) = 4.25$. So $f(0.2) = 5.2$ is the absolute maximum value and $f(1) = 2$ is the absolute minimum value.

58. $f(x) = \dfrac{x}{x^2 - x + 1},\ [0, 3]$.

$f'(x) = \dfrac{(x^2 - x + 1) - x(2x - 1)}{(x^2 - x + 1)^2} = \dfrac{x^2 - x + 1 - 2x^2 + x}{(x^2 - x + 1)^2} = \dfrac{1 - x^2}{(x^2 - x + 1)^2} = \dfrac{(1 + x)(1 - x)}{(x^2 - x + 1)^2} = 0 \quad \Leftrightarrow$

$x = \pm 1$, but $x = -1$ is not in the given interval, $[0, 3]$. $f(0) = 0$, $f(1) = 1$, and $f(3) = \frac{3}{7}$. So $f(1) = 1$ is the absolute maximum value and $f(0) = 0$ is the absolute minimum value.

59. $f(t) = t - \sqrt[3]{t},\ [-1, 4]$. $f'(t) = 1 - \frac{1}{3}t^{-2/3} = 1 - \dfrac{1}{3t^{2/3}}$. $f'(t) = 0 \quad \Leftrightarrow \quad 1 = \dfrac{1}{3t^{2/3}} \quad \Leftrightarrow \quad t^{2/3} = \dfrac{1}{3} \quad \Leftrightarrow$

$t = \pm\left(\dfrac{1}{3}\right)^{3/2} = \pm\sqrt{\dfrac{1}{27}} = \pm\dfrac{1}{3\sqrt{3}} = \pm\dfrac{\sqrt{3}}{9}$. $f'(t)$ does not exist when $t = 0$. $f(-1) = 0$, $f(0) = 0$,

$f\left(\dfrac{-1}{3\sqrt{3}}\right) = \dfrac{-1}{3\sqrt{3}} - \dfrac{-1}{\sqrt{3}} = \dfrac{-1 + 3}{3\sqrt{3}} = \dfrac{2\sqrt{3}}{9} \approx 0.3849$, $f\left(\dfrac{1}{3\sqrt{3}}\right) = \dfrac{1}{3\sqrt{3}} - \dfrac{1}{\sqrt{3}} = -\dfrac{2\sqrt{3}}{9}$, and

$f(4) = 4 - \sqrt[3]{4} \approx 2.413$. So $f(4) = 4 - \sqrt[3]{4}$ is the absolute maximum value and $f\left(\dfrac{\sqrt{3}}{9}\right) = -\dfrac{2\sqrt{3}}{9}$ is the absolute minimum value.

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60. $f(x)=\dfrac{e^x}{1+x^2}$, $[0,3]$. $f'(x)=\dfrac{(1+x^2)e^x-e^x(2x)}{(1+x^2)^2}=\dfrac{e^x(x^2-2x+1)}{(1+x^2)^2}=\dfrac{e^x(x-1)^2}{(1+x^2)^2}$. $f'(x)=0 \;\Rightarrow$ $(x-1)^2=0 \;\Leftrightarrow\; x=1$. $f'(x)$ exists for all real numbers since $1+x^2$ is never equal to 0. $f(0)=1$, $f(1)=e/2\approx 1.359$, and $f(3)=e^3/10\approx 2.009$. So $f(3)=e^3/10$ is the absolute maximum value and $f(0)=1$ is the absolute minimum value.

61. $f(t)=2\cos t+\sin 2t$, $[0,\pi/2]$.

$f'(t)=-2\sin t+\cos 2t\cdot 2=-2\sin t+2(1-2\sin^2 t)=-2(2\sin^2 t+\sin t-1)=-2(2\sin t-1)(\sin t+1)$.

$f'(t)=0 \;\Rightarrow\; \sin t=\frac{1}{2}$ or $\sin t=-1 \;\Rightarrow\; t=\frac{\pi}{6}$. $f(0)=2$, $f(\frac{\pi}{6})=\sqrt{3}+\frac{1}{2}\sqrt{3}=\frac{3}{2}\sqrt{3}\approx 2.60$, and $f(\frac{\pi}{2})=0$. So $f(\frac{\pi}{6})=\frac{3}{2}\sqrt{3}$ is the absolute maximum value and $f(\frac{\pi}{2})=0$ is the absolute minimum value.

62. $f(\theta)=1+\cos^2\theta$, $[\pi/4,\pi]$. $f'(\theta)=2\cos\theta(-\sin\theta)=-2\sin\theta\,\cos\theta=-\sin 2\theta$. $f'(\theta)=0 \;\Rightarrow\; -\sin 2\theta=0 \;\Rightarrow$ $2\theta=n\pi \;\Rightarrow\; \theta=\frac{n\pi}{2}$. Only $\theta=\frac{\pi}{2}$ $[n=1]$ is in the interval $(\pi/4,\pi)$. $f(\pi/4)=1+\left(\dfrac{\sqrt{2}}{2}\right)^2=\dfrac{3}{2}$, $f(\pi/2)=1+0^2=1$, and $f(\pi)=1+(-1)^2=2$. So $f(\pi)=2$ is the absolute maximum value and $f(\pi/2)=1$ is the absolute minimum value.

63. $f(x)=x^{-2}\ln x$, $\left[\frac{1}{2},4\right]$. $f'(x)=x^{-2}\cdot\dfrac{1}{x}+(\ln x)(-2x^{-3})=x^{-3}-2x^{-3}\ln x=x^{-3}(1-2\ln x)=\dfrac{1-2\ln x}{x^3}$.

$f'(x)=0 \;\Leftrightarrow\; 1-2\ln x=0 \;\Leftrightarrow\; 2\ln x=1 \;\Leftrightarrow\; \ln x=\frac{1}{2} \;\Leftrightarrow\; x=e^{1/2}\approx 1.65$. $f'(x)$ does not exist when $x=0$, which is not in the given interval, $\left[\frac{1}{2},4\right]$. $f(\frac{1}{2})=\dfrac{\ln 1/2}{(1/2)^2}=\dfrac{\ln 1-\ln 2}{1/4}=-4\ln 2\approx -2.773$, $f(e^{1/2})=\dfrac{\ln e^{1/2}}{(e^{1/2})^2}=\dfrac{1/2}{e}=\dfrac{1}{2e}\approx 0.184$, and $f(4)=\dfrac{\ln 4}{4^2}=\dfrac{\ln 4}{16}\approx 0.087$. So $f(e^{1/2})=\dfrac{1}{2e}$ is the absolute maximum value and $f\left(\frac{1}{2}\right)=-4\ln 2$ is the absolute minimum value.

64. $f(x)=xe^{x/2}$, $[-3,1]$. $f'(x)=xe^{x/2}\left(\frac{1}{2}\right)+e^{x/2}(1)=e^{x/2}\left(\frac{1}{2}x+1\right)$. $f'(x)=0 \;\Leftrightarrow\; \frac{1}{2}x+1=0 \;\Leftrightarrow\; x=-2$. $f(-3)=-3e^{-3/2}\approx -0.669$, $f(-2)=-2e^{-1}\approx -0.736$, and $f(1)=e^{1/2}\approx 1.649$. So $f(1)=e^{1/2}$ is the absolute maximum value and $f(-2)=-2/e$ is the absolute minimum value.

65. $f(x)=\ln(x^2+x+1)$, $[-1,1]$. $f'(x)=\dfrac{1}{x^2+x+1}\cdot(2x+1)=0 \;\Leftrightarrow\; x=-\frac{1}{2}$. Since $x^2+x+1>0$ for all x, the domain of f and f' is $\mathbb{R}$. $f(-1)=\ln 1=0$, $f\left(-\frac{1}{2}\right)=\ln\frac{3}{4}\approx -0.29$, and $f(1)=\ln 3\approx 1.10$. So $f(1)=\ln 3\approx 1.10$ is the absolute maximum value and $f\left(-\frac{1}{2}\right)=\ln\frac{3}{4}\approx -0.29$ is the absolute minimum value.

66. $f(x)=x-2\tan^{-1}x$, $[0,4]$. $f'(x)=1-2\cdot\dfrac{1}{1+x^2}=0 \;\Leftrightarrow\; 1=\dfrac{2}{1+x^2} \;\Leftrightarrow\; 1+x^2=2 \;\Leftrightarrow\; x^2=1 \;\Leftrightarrow$ $x=\pm 1$. $f(0)=0$, $f(1)=1-\frac{\pi}{2}\approx -0.57$, and $f(4)=4-2\tan^{-1}4\approx 1.35$. So $f(4)=4-2\tan^{-1}4$ is the absolute maximum value and $f(1)=1-\frac{\pi}{2}$ is the absolute minimum value.

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67. $f(x) = x^a(1-x)^b,\ \ 0 \le x \le 1, a > 0, b > 0.$

$f'(x) = x^a \cdot b(1-x)^{b-1}(-1) + (1-x)^b \cdot ax^{a-1} = x^{a-1}(1-x)^{b-1}[x \cdot b(-1) + (1-x) \cdot a]$

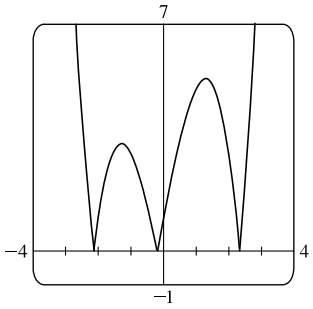
$= x^{a-1}(1-x)^{b-1}(a - ax - bx)$

At the endpoints, we have $f(0) = f(1) = 0$ [the minimum value of f]. In the interval $(0,1)$, $f'(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{a}{a+b}$.

$$f\left(\frac{a}{a+b}\right) = \left(\frac{a}{a+b}\right)^a \left(1 - \frac{a}{a+b}\right)^b = \frac{a^a}{(a+b)^a}\left(\frac{a+b-a}{a+b}\right)^b = \frac{a^a}{(a+b)^a} \cdot \frac{b^b}{(a+b)^b} = \frac{a^a b^b}{(a+b)^{a+b}}.$$

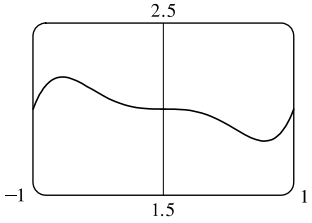
So $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the absolute maximum value.

68.



The graph of $f(x) = \left|1 + 5x - x^3\right|$ indicates that $f'(x) = 0$ at $x \approx \pm 1.3$ and that $f'(x)$ does not exist at $x \approx -2.1$, -0.2, and 2.3. Those five values of x are the critical numbers of f.

69. (a)



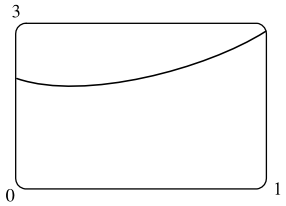
From the graph, it appears that the absolute maximum value is about $f(-0.77) = 2.19$, and the absolute minimum value is about $f(0.77) = 1.81$.

(b) $f(x) = x^5 - x^3 + 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 3x^2 = x^2(5x^2 - 3)$. So $f'(x) = 0 \quad \Rightarrow \quad x = 0, \pm\sqrt{\frac{3}{5}}$.

$$f\left(-\sqrt{\tfrac{3}{5}}\right) = \left(-\sqrt{\tfrac{3}{5}}\right)^5 - \left(-\sqrt{\tfrac{3}{5}}\right)^3 + 2 = -\left(\tfrac{3}{5}\right)^2\sqrt{\tfrac{3}{5}} + \tfrac{3}{5}\sqrt{\tfrac{3}{5}} + 2$$
$$= \left(\tfrac{3}{5} - \tfrac{9}{25}\right)\sqrt{\tfrac{3}{5}} + 2 = \tfrac{6}{25}\sqrt{\tfrac{3}{5}} + 2 \quad \text{(maximum)}$$

and similarly, $f\left(\sqrt{\frac{3}{5}}\right) = -\frac{6}{25}\sqrt{\frac{3}{5}} + 2$ (minimum).

70. (a)



From the graph, it appears that the absolute maximum value is about $f(1) = 2.85$, and the absolute minimum value is about $f(0.23) = 1.89$.

(b) $f(x) = e^x + e^{-2x} \quad \Rightarrow \quad f'(x) = e^x - 2e^{-2x} = e^{-2x}(e^{3x} - 2)$. So $f'(x) = 0 \quad \Leftrightarrow \quad e^{3x} = 2 \quad \Leftrightarrow \quad 3x = \ln 2 \quad \Leftrightarrow$ $x = \frac{1}{3}\ln 2$ [≈ 0.23]. $f\left(\frac{1}{3}\ln 2\right) = (e^{\ln 2})^{1/3} + (e^{\ln 2})^{-2/3} = 2^{1/3} + 2^{-2/3}$ [≈ 1.89], the minimum value.

$f(1) = e^1 + e^{-2}$ [≈ 2.85], the maximum.

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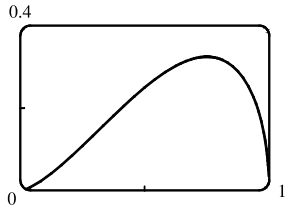
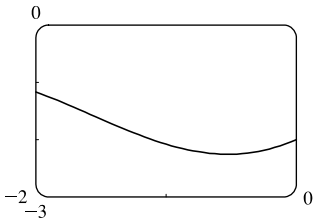
71. (a) 

From the graph, it appears that the absolute maximum value is about $f(0.75) = 0.32$, and the absolute minimum value is $f(0) = f(1) = 0$; that is, at both endpoints.

(b) $f(x) = x\sqrt{x - x^2} \quad \Rightarrow \quad f'(x) = x \cdot \dfrac{1 - 2x}{2\sqrt{x - x^2}} + \sqrt{x - x^2} = \dfrac{(x - 2x^2) + (2x - 2x^2)}{2\sqrt{x - x^2}} = \dfrac{3x - 4x^2}{2\sqrt{x - x^2}}$.

So $f'(x) = 0 \quad \Rightarrow \quad 3x - 4x^2 = 0 \quad \Rightarrow \quad x(3 - 4x) = 0 \quad \Rightarrow \quad x = 0$ or $\frac{3}{4}$.

$f(0) = f(1) = 0$ (minimum), and $f\left(\frac{3}{4}\right) = \frac{3}{4}\sqrt{\frac{3}{4} - \left(\frac{3}{4}\right)^2} = \frac{3}{4}\sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$ (maximum).

72. (a) 

From the graph, it appears that the absolute maximum value is about $f(-2) = -1.17$, and the absolute minimum value is about $f(-0.52) = -2.26$.

(b) $f(x) = x - 2\cos x \quad \Rightarrow \quad f'(x) = 1 + 2\sin x$. So $f'(x) = 0 \quad \Rightarrow \quad \sin x = -\frac{1}{2} \quad \Rightarrow \quad x = -\frac{\pi}{6}$ on $[-2, 0]$.

$f(-2) = -2 - 2\cos(-2)$ (maximum) and $f\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\cos\left(-\frac{\pi}{6}\right) = -\frac{\pi}{6} - 2\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6} - \sqrt{3}$ (minimum).

73. Let $a = 0.135$ and $b = -2.802$. Then $C(t) = ate^{bt} \quad \Rightarrow \quad C'(t) = a(t \cdot e^{bt} \cdot b + e^{bt} \cdot 1) = ae^{bt}(bt + 1)$. $C'(t) = 0 \quad \Leftrightarrow$

$bt + 1 = 0 \quad \Leftrightarrow \quad t = -\dfrac{1}{b} \approx 0.36$ h. $C(0) = 0$, $C(-1/b) = -\dfrac{a}{b}e^{-1} = -\dfrac{a}{be} \approx 0.0177$, and $C(3) = 3ae^{3b} \approx 0.00009$.

The maximum average BAC during the first three hours is about 0.0177 g/dL and it occurs at approximately 0.36 h $(21.4$ min$)$.

74. $C(t) = 8(e^{-0.4t} - e^{-0.6t}) \quad \Rightarrow \quad C'(t) = 8(-0.4e^{-0.4t} + 0.6e^{-0.6t})$. $C'(t) = 0 \quad \Leftrightarrow \quad 0.6e^{-0.6t} = 0.4e^{-0.4t} \quad \Leftrightarrow$

$\frac{0.6}{0.4} = e^{-0.4t + 0.6t} \quad \Leftrightarrow \quad \frac{3}{2} = e^{0.2t} \quad \Leftrightarrow \quad 0.2t = \ln\frac{3}{2} \quad \Leftrightarrow \quad t = 5\ln\frac{3}{2} \approx 2.027$ h. $C(0) = 8(1 - 1) = 0$,

$C\left(5\ln\frac{3}{2}\right) = 8(e^{-2\ln 3/2} - e^{-3\ln 3/2}) = 8\left[\left(\frac{3}{2}\right)^{-2} - \left(\frac{3}{2}\right)^{-3}\right] = 8\left(\frac{4}{9} - \frac{8}{27}\right) = \frac{32}{27} \approx 1.185$, and

$C(12) = 8(e^{-4.8} - e^{-7.2}) \approx 0.060$. The maximum concentration of the antibiotic during the first 12 hours is $\frac{32}{27}$ μg/mL.

75. The density is defined as $\rho = \dfrac{\text{mass}}{\text{volume}} = \dfrac{1000}{V(T)}$ (in g/cm^3). But a critical point of ρ will also be a critical point of V

$\left[\text{since } \dfrac{d\rho}{dT} = -1000V^{-2}\dfrac{dV}{dT} \text{ and } V \text{ is never } 0\right]$, and V is easier to differentiate than ρ.

$V(T) = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3 \quad \Rightarrow \quad V'(T) = -0.06426 + 0.0170086T - 0.0002037T^2$.

Setting this equal to 0 and using the quadratic formula to find T, we get

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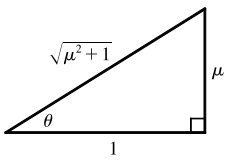
$T = \dfrac{-0.0170086 \pm \sqrt{0.0170086^2 - 4 \cdot 0.0002037 \cdot 0.06426}}{2(-0.0002037)} \approx 3.9665^\circ\text{C}$ or 79.5318°C. Since we are only interested in the region $0^\circ\text{C} \le T \le 30^\circ\text{C}$, we check the density ρ at the endpoints and at 3.9665°C: $\rho(0) \approx \dfrac{1000}{999.87} \approx 1.00013$; $\rho(30) \approx \dfrac{1000}{1003.7628} \approx 0.99625$; $\rho(3.9665) \approx \dfrac{1000}{999.7447} \approx 1.000255$. So water has its maximum density at about 3.9665°C.

76. $F = \dfrac{\mu W}{\mu \sin\theta + \cos\theta} \quad \Rightarrow \quad \dfrac{dF}{d\theta} = \dfrac{(\mu\sin\theta + \cos\theta)(0) - \mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2} = \dfrac{-\mu W(\mu\cos\theta - \sin\theta)}{(\mu\sin\theta + \cos\theta)^2}$.

So $\dfrac{dF}{d\theta} = 0 \quad \Rightarrow \quad \mu\cos\theta - \sin\theta = 0 \quad \Rightarrow \quad \mu = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$. Substituting $\tan\theta$ for μ in F gives us

$$F = \frac{(\tan\theta)W}{(\tan\theta)\sin\theta + \cos\theta} = \frac{W\tan\theta}{\dfrac{\sin^2\theta}{\cos\theta} + \cos\theta} = \frac{W\tan\theta\,\cos\theta}{\sin^2\theta + \cos^2\theta} = \frac{W\sin\theta}{1} = W\sin\theta.$$



If $\tan\theta = \mu$, then $\sin\theta = \dfrac{\mu}{\sqrt{\mu^2+1}}$ (see the figure), so $F = \dfrac{\mu}{\sqrt{\mu^2+1}}W$.

We compare this with the value of F at the endpoints: $F(0) = \mu W$ and $F\left(\frac{\pi}{2}\right) = W$.

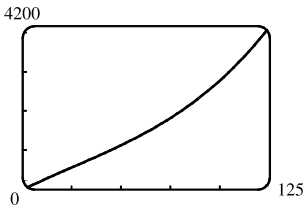
Now because $\dfrac{\mu}{\sqrt{\mu^2+1}} \le 1$ and $\dfrac{\mu}{\sqrt{\mu^2+1}} \le \mu$, we have that $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is less than or equal to each of $F(0)$ and $F\left(\frac{\pi}{2}\right)$.

Hence, $\dfrac{\mu}{\sqrt{\mu^2+1}}W$ is the absolute minimum value of $F(\theta)$, and it occurs when $\tan\theta = \mu$.

77. $L(t) = 0.01441t^3 - 0.4177t^2 + 2.703t + 1060.1 \quad \Rightarrow \quad L'(t) = 0.04323t^2 - 0.8354t + 2.703$. Use the quadratic formula to solve $L'(t) = 0$. $t = \dfrac{0.8354 \pm \sqrt{(0.8354)^2 - 4(0.04323)(2.703)}}{2(0.04323)} \approx 4.1$ or 15.2. For $0 \le t \le 12$, we have $L(0) = 1060.1$, $L(4.1) \approx 1065.2$, and $L(12) \approx 1057.3$. Thus, the water level was highest during 2012 about 4.1 months after January 1.

78. (a) The equation of the graph in the figure is

$$v(t) = 0.00146t^3 - 0.11553t^2 + 24.98169t - 21.26872.$$



(b) $a(t) = v'(t) = 0.00438t^2 - 0.23106t + 24.98169 \quad \Rightarrow$

$a'(t) = 0.00876t - 0.23106$.

$a'(t) = 0 \quad \Rightarrow \quad t_1 = \frac{0.23106}{0.00876} \approx 26.4$. $a(0) \approx 24.98$, $a(t_1) \approx 21.93$, and $a(125) \approx 64.54$.

The maximum acceleration is about 64.5 ft/s^2 and the minimum acceleration is about 21.93 ft/s^2.

79. (a) $v(r) = k(r_0 - r)r^2 = kr_0r^2 - kr^3 \quad \Rightarrow \quad v'(r) = 2kr_0r - 3kr^2$. $v'(r) = 0 \quad \Rightarrow \quad kr(2r_0 - 3r) = 0 \quad \Rightarrow$ $r = 0$ or $\frac{2}{3}r_0$ (but 0 is not in the interval). Evaluating v at $\frac{1}{2}r_0$, $\frac{2}{3}r_0$, and r_0, we get $v\left(\frac{1}{2}r_0\right) = \frac{1}{8}kr_0^3$, $v\left(\frac{2}{3}r_0\right) = \frac{4}{27}kr_0^3$, and $v(r_0) = 0$. Since $\frac{4}{27} > \frac{1}{8}$, v attains its maximum value at $r = \frac{2}{3}r_0$. This supports the statement in the text.

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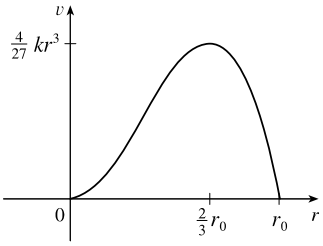
(b) From part (a), the maximum value of v is $\frac{4}{27}kr_0^3$.

(c)



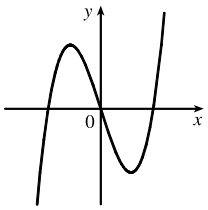
80. $f(x) = x^{101} + x^{51} + x + 1 \quad \Rightarrow \quad f'(x) = 101x^{100} + 51x^{50} + 1$. Since $f'(x) \geq 1$ for all x, $f'(x) = 0$ has no solution. Thus, f has no critical number, and the function f has neither a local maximum nor a local minimum.

81. (a) Suppose that f has a local minimum value at c, so $f(x) \geq f(c)$ for all x near c. Then $g(x) = -f(x) \leq -f(c) = g(c)$ for all x near c, so $g(x)$ has a local maximum value at c.

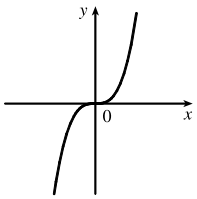
(b) If f has a local minimum at c, then $g(x) = -f(x)$ has a local maximum at c, so $g'(c) = 0$ by the case of Fermat's Theorem proved in the text. Thus, $f'(c) = -g'(c) = 0$.

82. (a) $f(x) = ax^3 + bx^2 + cx + d$, where $a \neq 0$. So $f'(x) = 3ax^2 + 2bx + c$ is a quadratic function and hence, the quadratic equation $f'(x) = 0$ has either 2, 1, or 0 real solutions. Thus, a cubic function can have two, one, or no critical number(s).

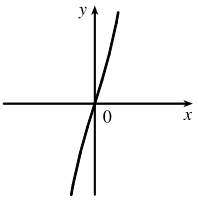
Case (i) [2 critical numbers]: $f(x) = x^3 - 3x \quad \Rightarrow$
$f'(x) = 3x^2 - 3 = 3(x^2 - 1)$, so $x = -1, 1$ are critical numbers.



Case (ii) [1 critical number]: $f(x) = x^3 \quad \Rightarrow$
$f'(x) = 3x^2$, so $x = 0$ is the only critical number.



Case (iii) [no critical number]: $f(x) = x^3 + 3x \quad \Rightarrow$
$f'(x) = 3x^2 + 3 = 3(x^2 + 1)$, so there is no critical number.



(b) Since there are at most two critical numbers, a cubic function can have at most two local extreme values, and by (a)(i), this can occur. By (a)(ii) and (a)(iii), it can have no local extreme value. Thus, a cubic function can have zero or two local extreme values.

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APPLIED PROJECT The Calculus of Rainbows

1. From Snell's Law, we have $\sin\alpha = k\sin\beta \approx \frac{4}{3}\sin\beta \quad\Leftrightarrow\quad \beta \approx \arcsin\left(\frac{3}{4}\sin\alpha\right)$. We substitute this into $D(\alpha) = \pi + 2\alpha - 4\beta = \pi + 2\alpha - 4\arcsin\left(\frac{3}{4}\sin\alpha\right)$, and then differentiate to find the minimum:

$D'(\alpha) = 2 - 4\left[1 - \left(\frac{3}{4}\sin\alpha\right)^2\right]^{-1/2}\left(\frac{3}{4}\cos\alpha\right) = 2 - \dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}}$. This is 0 when $\dfrac{3\cos\alpha}{\sqrt{1 - \frac{9}{16}\sin^2\alpha}} = 2 \quad\Leftrightarrow$

$\frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\sin^2\alpha \quad\Leftrightarrow\quad \frac{9}{4}\cos^2\alpha = 1 - \frac{9}{16}\left(1 - \cos^2\alpha\right) \quad\Leftrightarrow\quad \frac{27}{16}\cos^2\alpha = \frac{7}{16} \quad\Leftrightarrow\quad \cos\alpha = \sqrt{\frac{7}{27}} \quad\Leftrightarrow$

$\alpha = \arccos\sqrt{\frac{7}{27}} \approx 59.4^\circ$, and so the local minimum is $D(59.4^\circ) \approx 2.4$ radians $\approx 138^\circ$.

To see that this is an absolute minimum, we check the endpoints, which in this case are $\alpha = 0$ and $\alpha = \frac{\pi}{2}$:

$D(0) = \pi$ radians $= 180^\circ$, and $D\left(\frac{\pi}{2}\right) \approx 166^\circ$.

Another method: We first calculate $\dfrac{d\beta}{d\alpha}$: $\sin\alpha = \frac{4}{3}\sin\beta \quad\Leftrightarrow\quad \cos\alpha = \frac{4}{3}\cos\beta\,\dfrac{d\beta}{d\alpha} \quad\Leftrightarrow\quad \dfrac{d\beta}{d\alpha} = \dfrac{3}{4}\dfrac{\cos\alpha}{\cos\beta}$, so since

$D'(\alpha) = 2 - 4\,\dfrac{d\beta}{d\alpha} = 0 \quad\Leftrightarrow\quad \dfrac{d\beta}{d\alpha} = \dfrac{1}{2}$, the minimum occurs when $3\cos\alpha = 2\cos\beta$. Now we square both sides and substitute $\sin\alpha = \frac{4}{3}\sin\beta$, leading to the same result.

2. If we repeat Problem 1 with k in place of $\frac{4}{3}$, we get $D(\alpha) = \pi + 2\alpha - 4\arcsin\left(\dfrac{1}{k}\sin\alpha\right) \quad\Rightarrow$

$D'(\alpha) = 2 - \dfrac{4\cos\alpha}{k\sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2}}$, which is 0 when $\dfrac{2\cos\alpha}{k} = \sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2} \quad\Leftrightarrow\quad \left(\dfrac{2\cos\alpha}{k}\right)^2 = 1 - \left(\dfrac{\sin\alpha}{k}\right)^2 \quad\Leftrightarrow$

$4\cos^2\alpha = k^2 - \sin^2\alpha \quad\Leftrightarrow\quad 3\cos^2\alpha = k^2 - 1 \quad\Leftrightarrow\quad \alpha = \arccos\sqrt{\dfrac{k^2 - 1}{3}}$. So for $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx 1.038$ radians, and so the rainbow angle is about $\pi - D(\alpha_1) \approx 42.3^\circ$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx 1.026$ radians, and so the rainbow angle is about $\pi - D(\alpha_2) \approx 40.6^\circ$.

Another method: As in Problem 1, we can instead find $D'(\alpha)$ in terms of $\dfrac{d\beta}{d\alpha}$, and then substitute $\dfrac{d\beta}{d\alpha} = \dfrac{\cos\alpha}{k\cos\beta}$.

3. At each reflection or refraction, the light is bent in a counterclockwise direction: the bend at A is $\alpha - \beta$, the bend at B is $\pi - 2\beta$, the bend at C is again $\pi - 2\beta$, and the bend at D is $\alpha - \beta$. So the total bend is $D(\alpha) = 2(\alpha - \beta) + 2(\pi - 2\beta) = 2\alpha - 6\beta + 2\pi$, as required. We substitute $\beta = \arcsin\left(\dfrac{\sin\alpha}{k}\right)$ and differentiate, to get

$D'(\alpha) = 2 - \dfrac{6\cos\alpha}{k\sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2}}$, which is 0 when $\dfrac{3\cos\alpha}{k} = \sqrt{1 - \left(\dfrac{\sin\alpha}{k}\right)^2} \quad\Leftrightarrow\quad 9\cos^2\alpha = k^2 - \sin^2\alpha \quad\Leftrightarrow$

$8\cos^2\alpha = k^2 - 1 \quad\Leftrightarrow\quad \cos\alpha = \sqrt{\frac{1}{8}(k^2 - 1)}$. If $k = \frac{4}{3}$, then the minimum occurs at

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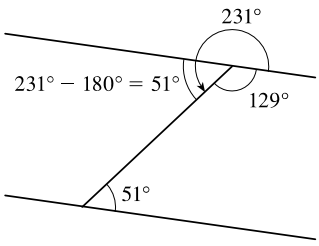
$\alpha_1 = \arccos\sqrt{\dfrac{(4/3)^2 - 1}{8}} \approx 1.254$ radians. Thus, the minimum *counterclockwise* rotation is $D(\alpha_1) \approx 231°$, which is equivalent to a *clockwise* rotation of $360° - 231° = 129°$ (see the figure). So the rainbow angle for the secondary rainbow is about $180° - 129° = 51°$, as required. In general, the rainbow angle for the secondary rainbow is $\pi - [2\pi - D(\alpha)] = D(\alpha) - \pi$.



4. In the primary rainbow, the rainbow angle gets smaller as k gets larger, as we found in Problem 2, so the colors appear from top to bottom in order of increasing k. But in the secondary rainbow, the rainbow angle gets larger as k gets larger. To see this, we find the minimum deviations for red light and for violet light in the secondary rainbow. For $k \approx 1.3318$ (red light) the minimum occurs at $\alpha_1 \approx \arccos\sqrt{\dfrac{1.3318^2 - 1}{8}} \approx 1.255$ radians, and so the rainbow angle is $D(\alpha_1) - \pi \approx 50.6°$. For $k \approx 1.3435$ (violet light) the minimum occurs at $\alpha_2 \approx \arccos\sqrt{\dfrac{1.3435^2 - 1}{8}} \approx 1.248$ radians, and so the rainbow angle is $D(\alpha_2) - \pi \approx 53.6°$. Consequently, the rainbow angle is larger for colors with higher indices of refraction, and the colors appear from bottom to top in order of increasing k, the reverse of their order in the primary rainbow.

Note that our calculations above also explain why the secondary rainbow is more spread out than the primary rainbow: in the primary rainbow, the difference between rainbow angles for red and violet light is about $1.7°$, whereas in the secondary rainbow it is about $3°$.

4.2 The Mean Value Theorem

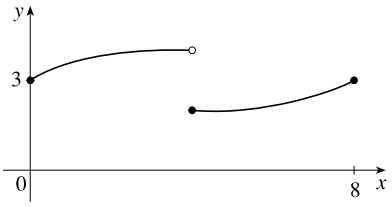
1. **(1)** f is continuous on the closed interval $[0, 8]$.

(2) f is differentiable on the open interval $(0, 8)$.

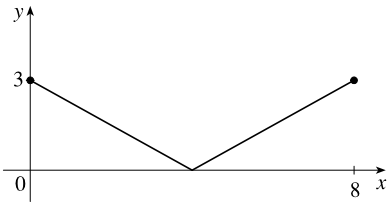
(3) $f(0) = 3$ and $f(8) = 3$

Thus, f satisfies the hypotheses of Rolle's Theorem. The numbers $c = 1$ and $c = 5$ satisfy the conclusion of Rolle's Theorem since $f'(1) = f'(5) = 0$.

2. The possible graphs fall into two general categories: **(1)** Not continuous and therefore not differentiable, **(2)** Continuous, but not differentiable.



Not continuous



Not differentiable

In either case, there is no number c such that $f'(c) = 0$.

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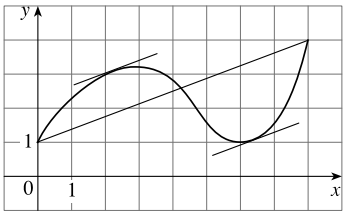
3. (a) **(1)** g is continuous on the closed interval $[0, 8]$.

(2) g is differentiable on the open interval $(0, 8)$.

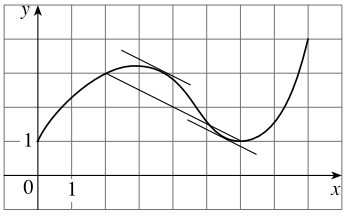
(b) $g'(c) = \dfrac{g(8) - g(0)}{8 - 0} = \dfrac{4 - 1}{8} = \dfrac{3}{8}$.

It appears that $g'(c) = \frac{3}{8}$ when $c \approx 2.2$ and 6.4.

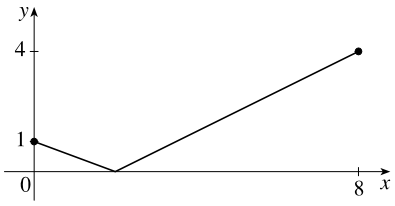


(c) $g'(c) = \dfrac{g(6) - g(2)}{6 - 2} = \dfrac{1 - 3}{4} = -\dfrac{1}{2}$.

It appears that $g'(c) = -\frac{1}{2}$ when $c \approx 3.7$ and 5.5.



4.



The function shown in the figure is continuous on $[0, 8]$ [but not differentiable on $(0, 8)$] with $f(0) = 1$ and $f(8) = 4$. The line passing through the two points has slope $\frac{3}{8}$. There is no number c in $(0, 8)$ such that $f'(c) = \frac{3}{8}$.

5. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is not differentiable on the open interval $(0, 5)$ since f is not differentiable at 3.

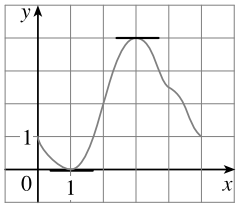
Thus, f does not satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

6. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is differentiable on the open interval $(0, 5)$.

Thus, f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

The line passing through $(0, f(0))$ and $(5, f(5))$ has slope 0. It appears that $f'(c) = 0$ for $c = 1$ and $c = 3$.

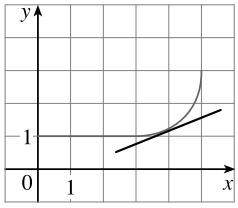


7. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is differentiable on the open interval $(0, 5)$.

Thus, f satisfies the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

The line passing through $(0, f(0))$ and $(5, f(5))$ has slope $\dfrac{f(5) - f(0)}{5 - 0} = \dfrac{3 - 1}{5} = \dfrac{2}{5}$. It appears that $f'(c) = \frac{2}{5}$ when $c \approx 3.8$.



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8. **(1)** f is continuous on the closed interval $[0, 5]$.

(2) f is not differentiable on the open interval $(0, 5)$ since f is not differentiable at 4.

Thus, f does not satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 5]$.

9. $f(x) = 2x^2 - 4x + 5,\ [-1, 3]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-1, 3]$ and differentiable on $(-1, 3)$. Since $f(-1) = 11$ and $f(3) = 11$, f satisfies all the hypotheses of Rolle's Theorem. $f'(c) = 4c - 4$ and $f'(c) = 0 \quad \Leftrightarrow \quad 4c - 4 = 0 \quad \Leftrightarrow \quad c = 1$. $c = 1$ is in the interval $(-1, 3)$, so 1 satisfies the conclusion of Rolle's Theorem.

10. $f(x) = x^3 - 2x^2 - 4x + 2,\ [-2, 2]$. f is a polynomial, so it's continuous and differentiable on $\mathbb{R}$, and hence, continuous on $[-2, 2]$ and differentiable on $(-2, 2)$. Since $f(-2) = -6$ and $f(2) = -6$, f satisfies all the hypotheses of Rolle's Theorem. $f'(c) = 3c^2 - 4c - 4$ and $f'(c) = 0 \quad \Leftrightarrow \quad (3c + 2)(c - 2) = 0 \quad \Leftrightarrow \quad c = -\frac{2}{3}$ or 2. $c = -\frac{2}{3}$ is in the open interval $(-2, 2)$ (but 2 *isn't*), so only $-\frac{2}{3}$ satisfies the conclusion of Rolle's Theorem.

11. $f(x) = \sin(x/2),\ [\pi/2, 3\pi/2]$. f, being the composite of the sine function and the polynomial $x/2$, is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[\pi/2, 3\pi/2]$ and differentiable on $(\pi/2, 3\pi/2)$. Also, $f\left(\frac{\pi}{2}\right) = \frac{1}{2}\sqrt{2} = f\left(\frac{3\pi}{2}\right)$.
$f'(c) = 0 \quad \Leftrightarrow \quad \frac{1}{2}\cos(c/2) = 0 \quad \Leftrightarrow \quad \cos(c/2) = 0 \quad \Leftrightarrow \quad c/2 = \frac{\pi}{2} + n\pi$ [n an integer] $\quad \Leftrightarrow \quad c = \pi + 2n\pi$.
Only $c = \pi$ [when $n = 0$] is in $(\pi/2, 3\pi/2)$, so π satisfies the conclusion of Rolle's Theorem.

12. $f(x) = x + 1/x,\ \left[\frac{1}{2}, 2\right]$. $f'(x) = 1 - 1/x^2 = \dfrac{x^2 - 1}{x^2}$. f is a rational function that is continuous on its domain, $(-\infty, 0) \cup (0, \infty)$, so it is continuous on $\left[\frac{1}{2}, 2\right]$. f' has the same domain and is differentiable on $\left(\frac{1}{2}, 2\right)$. Also, $f\left(\frac{1}{2}\right) = \frac{5}{2} = f(2)$. $f'(c) = 0 \quad \Leftrightarrow \quad \dfrac{c^2 - 1}{c^2} = 0 \quad \Leftrightarrow \quad c^2 - 1 = 0 \quad \Leftrightarrow \quad c = \pm 1$. Only 1 is in $\left(\frac{1}{2}, 2\right)$, so 1 satisfies the conclusion of Rolle's Theorem.

13. $f(x) = 1 - x^{2/3}$. $f(-1) = 1 - (-1)^{2/3} = 1 - 1 = 0 = f(1)$. $f'(x) = -\frac{2}{3}x^{-1/3}$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'(0)$ does not exist, and so f is not differentiable on $(-1, 1)$.

14. $f(x) = \tan x$. $f(0) = \tan 0 = 0 = \tan \pi = f(\pi)$. $f'(x) = \sec^2 x \geq 1$, so $f'(c) = 0$ has no solution. This does not contradict Rolle's Theorem, since $f'\left(\frac{\pi}{2}\right)$ does not exist, and so f is not differentiable on $(0, \pi)$. (Also, f is not continuous on $[0, \pi]$.)

15. $f(x) = 2x^2 - 3x + 1,\ [0, 2]$. f is continuous on $[0, 2]$ and differentiable on $(0, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 4c - 3 = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{3 - 1}{2} = 1 \quad \Leftrightarrow \quad 4c = 4 \quad \Leftrightarrow \quad c = 1$, which is in $(0, 2)$.

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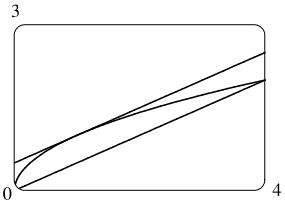
16. $f(x) = x^3 - 3x + 2,\ [-2, 2]$. f is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 3c^2 - 3 = \dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 0}{4} = 1 \quad \Leftrightarrow \quad 3c^2 = 4 \quad \Leftrightarrow$ $c^2 = \dfrac{4}{3} \quad \Leftrightarrow \quad c = \pm\dfrac{2}{\sqrt{3}} \approx \pm 1.15$, which are both in $(-2, 2)$.

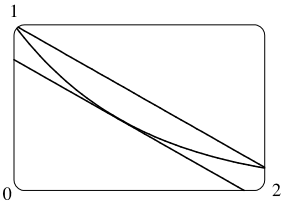
17. $f(x) = \ln x,\ [1, 4]$. f is continuous and differentiable on $(0, \infty)$, so f is continuous on $[1, 4]$ and differentiable on $(1, 4)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad \dfrac{1}{c} = \dfrac{f(4) - f(1)}{4 - 1} = \dfrac{\ln 4 - 0}{3} = \dfrac{\ln 4}{3} \quad \Leftrightarrow \quad c = \dfrac{3}{\ln 4} \approx 2.16$, which is in $(1, 4)$.

18. $f(x) = 1/x,\ [1, 3]$. f is continuous and differentiable on $(-\infty, 0) \cup (0, \infty)$, so f is continous on $[1, 3]$ and differentiable on $(1, 3)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad -\dfrac{1}{c^2} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{\frac{1}{3} - 1}{2} = -\dfrac{1}{3} \quad \Leftrightarrow \quad c^2 = 3 \quad \Leftrightarrow \quad c = \pm\sqrt{3} \approx \pm 1.73$, but only $\sqrt{3}$ is in $(1, 3)$.

19. $f(x) = \sqrt{x},\ [0, 4]$. $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad \Leftrightarrow \quad \dfrac{1}{2\sqrt{c}} = \dfrac{2 - 0}{4} \quad \Leftrightarrow$ $\dfrac{1}{2\sqrt{c}} = \dfrac{1}{2} \quad \Leftrightarrow \quad \sqrt{c} = 1 \quad \Leftrightarrow \quad c = 1$. The secant line and the tangent line are parallel.



20. $f(x) = e^{-x}, [0, 2]$. $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} \quad \Leftrightarrow \quad -e^{-c} = \dfrac{e^{-2} - 1}{2} \quad \Leftrightarrow$ $e^{-c} = \dfrac{1 - e^{-2}}{2} \quad \Leftrightarrow \quad -c = \ln \dfrac{1 - e^{-2}}{2} \quad \Leftrightarrow$ $c = -\ln \dfrac{1 - e^{-2}}{2} \approx 0.84$. The secant line and the tangent line are parallel.



21. $f(x) = (x - 3)^{-2} \quad \Rightarrow \quad f'(x) = -2(x - 3)^{-3}$. $f(4) - f(1) = f'(c)(4 - 1) \quad \Rightarrow \quad \dfrac{1}{1^2} - \dfrac{1}{(-2)^2} = \dfrac{-2}{(c - 3)^3} \cdot 3 \quad \Rightarrow$ $\dfrac{3}{4} = \dfrac{-6}{(c - 3)^3} \quad \Rightarrow \quad (c - 3)^3 = -8 \quad \Rightarrow \quad c - 3 = -2 \quad \Rightarrow \quad c = 1$, which is not in the *open* interval $(1, 4)$. This does not contradict the Mean Value Theorem since f is not continuous at $x = 3$, which is in the interval $[1, 4]$.

22. $f(x) = 2 - |2x - 1| = \begin{cases} 2 - (2x - 1) & \text{if } 2x - 1 \geq 0 \\ 2 - [-(2x - 1)] & \text{if } 2x - 1 < 0 \end{cases} = \begin{cases} 3 - 2x & \text{if } x \geq \frac{1}{2} \\ 1 + 2x & \text{if } x < \frac{1}{2} \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} -2 & \text{if } x > \frac{1}{2} \\ 2 & \text{if } x < \frac{1}{2} \end{cases}$

$f(3) - f(0) = f'(c)(3 - 0) \quad \Rightarrow \quad -3 - 1 = f'(c) \cdot 3 \quad \Rightarrow \quad f'(c) = -\frac{4}{3}$ [not ± 2]. This does not contradict the Mean Value Theorem since f is not differentiable at $x = \frac{1}{2}$, which is in the interval $(0, 3)$.

23. Let $f(x) = 2x + \cos x$. Then $f(-\pi) = -2\pi - 1 < 0$ and $f(0) = 1 > 0$. Since f is the sum of the polynomial $2x$ and the trignometric function $\cos x$, f is continuous and differentiable for all x. By the Intermediate Value Theorem, there is a

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number c in $(-\pi, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real solution. If the equation has distinct real solutions a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 2 - \sin r > 0$ since $\sin r \leq 1$. This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one solution.

24. Let $f(x) = x^3 + e^x$. Then $f(-1) = -1 + 1/e < 0$ and $f(0) = 1 > 0$. Since f is the sum of a polynomial and the natural exponential function, f is continous and differentiable for all x. By the Intermediate Value Theorem, there is a number c in $(-1, 0)$ such that $f(c) = 0$. Thus, the given equation has at least one real solution. If the equation has distinct real solutions a and b with $a < b$, then $f(a) = f(b) = 0$. Since f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. But $f'(r) = 3r^2 + e^r > 0$. This contradiction shows that the given equation can't have two distinct real solutions, so it has exactly one solution.

25. Let $f(x) = x^3 - 15x + c$ for x in $[-2, 2]$. If f has two real solutions a and b in $[-2, 2]$, with $a < b$, then $f(a) = f(b) = 0$. Since the polynomial f is continuous on $[a, b]$ and differentiable on (a, b), Rolle's Theorem implies that there is a number r in (a, b) such that $f'(r) = 0$. Now $f'(r) = 3r^2 - 15$. Since r is in (a, b), which is contained in $[-2, 2]$, we have $|r| < 2$, so $r^2 < 4$. It follows that $3r^2 - 15 < 3 \cdot 4 - 15 = -3 < 0$. This contradicts $f'(r) = 0$, so the given equation, $x^3 - 15x + c = 0$, can't have two real solutions in $[-2, 2]$. Hence, it has at most one real solution in $[-2, 2]$.

26. $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real solutions a, b, d where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers c_1 and c_2 with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)(x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, the given equation, $x^4 + 4x + c = 0$, can have at most two real solutions.

27. (a) Suppose that a cubic polynomial $P(x)$ has zeros $a_1 < a_2 < a_3 < a_4$, so $P(a_1) = P(a_2) = P(a_3) = P(a_4)$. By Rolle's Theorem there are numbers c_1, c_2, c_3 with $a_1 < c_1 < a_2$, $a_2 < c_2 < a_3$, and $a_3 < c_3 < a_4$ and $P'(c_1) = P'(c_2) = P'(c_3) = 0$. Thus, the second-degree polynomial $P'(x)$ has three distinct real zeros, which is impossible. This contradiction tells us that a polynomial of degree 3 has at most three real zeros.

(b) We prove by induction that a polynomial of degree n has at most n real zeros. This is certainly true for $n = 1$. Suppose that the result is true for all polynomials of degree n and let $P(x)$ be a polynomial of degree $n + 1$. Suppose that $P(x)$ has more than $n + 1$ real zeros, say $a_1 < a_2 < a_3 < \cdots < a_{n+1} < a_{n+2}$. Then $P(a_1) = P(a_2) = \cdots = P(a_{n+2}) = 0$. By Rolle's Theorem there are real numbers $c_1, \ldots, c_{n+1}$ with $a_1 < c_1 < a_2, \ldots, a_{n+1} < c_{n+1} < a_{n+2}$ and $P'(c_1) = \cdots = P'(c_{n+1}) = 0$. Thus, the nth-degree polynomial $P'(x)$ has at least $n + 1$ zeros. This contradiction shows that $P(x)$ has at most $n + 1$ real zeros and hence, a polynomial of degree n has at most n real zeros.

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28. (a) Suppose that $f(a) = f(b) = 0$ where $a < b$. By Rolle's Theorem applied to f on $[a, b]$, there is a number c such that $a < c < b$ and $f'(c) = 0$.

(b) Suppose that $f(a) = f(b) = f(c) = 0$ where $a < b < c$. By Rolle's Theorem applied to f on $[a, b]$ and $[b, c]$, there are numbers d and e such that $a < d < b$ and $b < e < c$, with $f'(d) = 0$ and $f'(e) = 0$. By Rolle's Theorem applied to f' on $[d, e]$, there is a number g such that $d < g < e$ and $f''(g) = 0$.

(c) Suppose that f is n times differentiable on $\mathbb{R}$ and has $n + 1$ distinct real zeros. Then $f^{(n)}$ has at least one real zero.

29. By the Mean Value Theorem, $f(4) - f(1) = f'(c)(4 - 1)$ for some $c \in (1, 4)$. (f is differentiable for all x, so, in particular, f is differentiable on $(1, 4)$ and continuous on $[1, 4]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) For every $c \in (1, 4)$, we have $f'(c) \geq 2$. Putting $f'(c) \geq 2$ into the above equation and substituting $f(1) = 10$, we get $f(4) = f(1) + f'(c)(4 - 1) = 10 + 3f'(c) \geq 10 + 3 \cdot 2 = 16$. So the smallest possible value of $f(4)$ is 16.

30. By the Mean Value Theorem, $f(8) - f(2) = f'(c)(8 - 2)$ for some $c \in (2, 8)$. (f is differentiable for all x, so, in particular, f is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that $6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5$, so $18 \leq f(8) - f(2) \leq 30$.

31. Suppose that such a function f exists. By the Mean Value Theorem, there is a number c such that $0 < c < 2$ with $f'(c) = \dfrac{f(2) - f(0)}{2 - 0} = \dfrac{4 - (-1)}{2 - 0} = \dfrac{5}{2}$. This result, $f'(c) = \dfrac{5}{2}$, is impossible since $f'(x) \leq 2$ for all x, so no such function f exists.

32. Let $h = f - g$. Note that since $f(a) = g(a)$, $h(a) = f(a) - g(a) = 0$. Then since f and g are continuous on $[a, b]$ and differentiable on (a, b), so is h, and thus h satisfies the assumptions of the Mean Value Theorem. Therefore, there is a number c with $a < c < b$ such that $h(b) = h(b) - h(a) = h'(c)(b - a)$. Given $f'(x) < g'(x)$, we have $f' - g' < 0$ or, equivalently, $h' < 0$. Now since $h'(c) < 0$, $h'(c)(b - a) < 0$, so $h(b) = f(b) - g(b) < 0$ and hence $f(b) < g(b)$.

33. Consider the function $f(x) = \sin x$, which is continuous and differentiable on $\mathbb{R}$. Let a be a number such that $0 < a < 2\pi$. Then f is continuous on $[0, a]$ and differentiable on $(0, a)$. By the Mean Value Theorem, there is a number c in $(0, a)$ such that $f(a) - f(0) = f'(c)(a - 0)$; that is, $\sin a - 0 = (\cos c)(a)$. Now $\cos c < 1$ for $0 < c < 2\pi$, so $\sin a < 1 \cdot a = a$. We took a to be an arbitrary number in $(0, 2\pi)$, so $\sin x < x$ for all x satisfying $0 < x < 2\pi$.

34. f satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $f'(c) = \dfrac{f(b) - f(-b)}{b - (-b)}$ for some $c \in (-b, b)$. Since f is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get $f'(c) = \dfrac{f(b) + f(b)}{2b}$, or, equivalently, $f'(c) = \dfrac{f(b)}{b}$.

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35. Let $f(x) = \sin x$ on $[a, b]$. Then f is continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem, there is a number $c \in (a, b)$ with $f(b) - f(a) = f'(c)(b - a)$ or, equivalently, $\sin b - \sin a = (\cos c)(b - a)$. Taking absolute values, $|\sin b - \sin a| \le |\cos c|\,|b - a|$ or, equivalently, $|\sin a - \sin b| \le 1\,|b - a|$. If $b < a$, then $|\sin a - \sin b| \le |a - b|$. If $a = b$, both sides of the inequality are 0, which proves the given inequality for all a and b.

36. Suppose that $f'(x) = c$. Let $g(x) = cx$, so $g'(x) = c$. Then, by Corollary 7, $f(x) = g(x) + d$, where d is a constant, so $f(x) = cx + d$.

37. For $x > 0$, $f(x) = g(x)$, so $f'(x) = g'(x)$. For $x < 0$, $f'(x) = (1/x)' = -1/x^2$ and $g'(x) = (1 + 1/x)' = -1/x^2$, so again $f'(x) = g'(x)$. However, the domain of $g(x)$ is not an interval [it is $(-\infty, 0) \cup (0, \infty)$], so we cannot conclude that $f - g$ is constant (in fact, it is not).

38. Let $f(x) = \arctan x + \arctan\left(\dfrac{1}{x}\right)$. Then $f'(x) = \dfrac{1}{1+x^2} + \dfrac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \left(-\dfrac{1}{x^2}\right) = \dfrac{1}{1+x^2} - \dfrac{1}{x^2+1} = 0$ on the given domain, $(0, \infty)$. Therefore, $f(x) = C$ on $(0, \infty)$ by Theorem 5. To find C, we let $x = 1$ to get $\arctan 1 + \arctan\left(\dfrac{1}{1}\right) = C \quad \Rightarrow \quad \dfrac{\pi}{4} + \dfrac{\pi}{4} = \dfrac{\pi}{2} = C$. Thus, $f(x) = \dfrac{\pi}{2}$; that is, $\arctan x + \arctan\left(\dfrac{1}{x}\right) = \dfrac{\pi}{2}$.

39. Let $f(x) = 2\sin^{-1} x - \cos^{-1}(1 - 2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \left(-\frac{-4x}{\sqrt{1-(1-2x^2)^2}}\right) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{4x^2-4x^4}}$$

$$= \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} \text{ [since } x \ge 0] \quad = 0$$

Thus, $f'(x) = 0$ for all $x \in (0, 1)$, and hence, $f(x) = C$ on $(0, 1)$. To find C, let $x = 0.5$ to get $f(0.5) = 2\,\sin^{-1}(0.5) - \cos^{-1}(0.5) = 2\left(\frac{\pi}{6}\right) - \frac{\pi}{3} = 0 = C$. We conclude that $f(x) = 0$ for x in $(0, 1)$. By continuity of f, $f(x) = 0$ on $[0, 1]$. Therefore, we see that $f(x) = 2\,\sin^{-1} x - \cos^{-1}(1 - 2x^2) = 0 \quad \Rightarrow \quad 2\,\sin^{-1} x = \cos^{-1}(1 - 2x^2)$.

40. Let $v(t)$ be the velocity of the car t hours after 2:00 PM. Then $\dfrac{v(1/6) - v(0)}{1/6 - 0} = \dfrac{50 - 30}{1/6} = 120$. By the Mean Value Theorem, there is a number c such that $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time t, the acceleration c hours after 2:00 PM is exactly 120 mi/h^2.

41. Let $g(t)$ and $h(t)$ be the position functions of the two runners and let $f(t) = g(t) - h(t)$. By hypothesis, where b is the finishing time, $f(0) = g(0) - h(0) = 0$ and $f(b) = g(b) - h(b) = 0$. Then by the Mean Value Theorem, there is a time c, with $0 < c < b$, such that $f'(c) = \dfrac{f(b) - f(0)}{b - 0} = \dfrac{0 - 0}{b} = 0$. Since $f'(c) = g'(c) - h'(c) = 0$, we have $g'(c) = h'(c)$ [the velocities are equal]. So at time c, both runners have the same speed.

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42. Assume that f is differentiable (and hence continuous) on $\mathbb{R}$ and that $f'(x) \neq 1$ for all x. Suppose f has more than one fixed point. Then there are numbers a and b such that $a < b$, $f(a) = a$, and $f(b) = b$. Applying the Mean Value Theorem to the function f on $[a, b]$, we find that there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. But then $f'(c) = \dfrac{b - a}{b - a} = 1$, contradicting our assumption that $f'(x) \neq 1$ for every real number x. This shows that our supposition was wrong, that is, that f cannot have more than one fixed point.

4.3 What Derivatives Tell Us about the Shape of a Graph

1. (a) f is increasing on $(1, 3)$ and $(4, 6)$.

(b) f is decreasing on $(0, 1)$ and $(3, 4)$.

(c) f is concave upward on $(0, 2)$.

(d) f is concave downward on $(2, 4)$ and $(4, 6)$.

(e) The point of inflection is $(2, 3)$.

2. (a) f is increasing on $(0, 1)$ and $(3, 7)$.

(b) f is decreasing on $(1, 3)$.

(c) f is concave upward on $(2, 4)$ and $(5, 7)$.

(d) f is concave downward on $(0, 2)$ and $(4, 5)$.

(e) The points of inflection are $(2, 2)$, $(4, 3)$, and $(5, 4)$.

3. (a) Use the Increasing/Decreasing (I/D) Test.

(b) Use the Concavity Test.

(c) At any value of x where the concavity changes, we have an inflection point at $(x, f(x))$.

4. (a) See the First Derivative Test.

(b) See the Second Derivative Test and the note that precedes Example 6.

5. (a) Since $f'(x) > 0$ on $(0, 1)$ and $(3, 5)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(1, 3)$ and $(5, 6)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$, and f' changes from positive to negative at both values, f changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$, and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 3$.

6. (a) Since $f'(x) > 0$ on $(1, 4)$ and $(5, 6)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(0, 1)$ and $(4, 5)$, f is decreasing on these intervals.

(b) Since $f'(x) = 0$ at $x = 4$, and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 4$. Since $f'(x) = 0$ at $x = 1$ and $x = 5$, and f' changes from negative to positive at both values, f changes from decreasing to increasing and has local minima at $x = 1$ and $x = 5$.

7. (a) There is an IP at $x = 3$ because the graph of f changes from CD to CU there. There is an IP at $x = 5$ because the graph of f changes from CU to CD there.

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(b) There is an IP at $x = 2$ and at $x = 6$ because $f'(x)$ has a maximum value there, and so $f''(x)$ changes from positive to negative there. There is an IP at $x = 4$ because $f'(x)$ has a minimum value there and so $f''(x)$ changes from negative to positive there.

(c) There is an inflection point at $x = 1$ because $f''(x)$ changes from negative to positive there, and so the graph of f changes from concave downward to concave upward. There is an inflection point at $x = 7$ because $f''(x)$ changes from positive to negative there, and so the graph of f changes from concave upward to concave downward.

8. (a) f is increasing when f' is positive. This happens on the intervals $(0, 4)$ and $(6, 8)$.

(b) f has a local maximum where it changes from increasing to decreasing, that is, where f' changes from positive to negative (at $x = 4$ and $x = 8$). Similarly, f has a local minimum where f' changes from negative to positive (at $x = 6$).

(c) f is concave upward where f' is increasing (hence f'' is positive). This happens on $(0, 1)$, $(2, 3)$, and $(5, 7)$. Similarly, f is concave downward where f' is decreasing, that is, on $(1, 2)$, $(3, 5)$, and $(7, 9)$.

(d) f has an inflection point where the concavity changes. This happens at $x = 1$, 2, 3, 5, and 7.

9. $f(x) = 2x^3 - 15x^2 + 24x - 5 \quad \Rightarrow \quad f'(x) = 6x^2 - 30x + 24 = 6(x^2 - 5x + 4) = 6(x - 1)(x - 4)$.

Interval	$x - 1$	$x - 4$	$f'(x)$	f
$x < 1$	$-$	$-$	$+$	increasing on $(-\infty, 1)$
$1 < x < 4$	$+$	$-$	$-$	decreasing on $(1, 4)$
$x > 4$	$+$	$+$	$+$	increasing on $(4, \infty)$

f changes from increasing to decreasing at $x = 1$ and from decreasing to increasing at $x = 4$. Thus, $f(1) = 6$ is a local maximum value and $f(4) = -21$ is a local minimum value.

10. $f(x) = x^3 - 6x^2 - 135x \quad \Rightarrow \quad f'(x) = 3x^2 - 12x - 135 = 3(x^2 - 4x - 45) = 3(x + 5)(x - 9)$.

Interval	$x + 5$	$x - 9$	$f'(x)$	f
$x < -5$	$-$	$-$	$+$	increasing on $(-\infty, -5)$
$-5 < x < 9$	$+$	$-$	$-$	decreasing on $(-5, 9)$
$x > 9$	$+$	$+$	$+$	increasing on $(9, \infty)$

f changes from increasing to decreasing at $x = -5$ and from decreasing to increasing at $x = 9$. Thus, $f(-5) = 400$ is a local maximum value and $f(9) = -972$ is a local minimum value.

11. $f(x) = 6x^4 - 16x^3 + 1 \quad \Rightarrow \quad f'(x) = 24x^3 - 48x^2 = 24x^2(x - 2)$.

Interval	x^2	$x - 2$	$f'(x)$	f
$x < 0$	$+$	$-$	$-$	decreasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

Note that f is differentiable and $f'(x) < 0$ on the interval $(-\infty, 2)$ except for the single number $x = 0$. By applying the result

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of Exercise 4.3.97, we can say that f is decreasing on the entire interval $(-\infty, 2)$. f changes from decreasing to increasing at $x = 2$. Thus, $f(2) = -31$ is a local minimum value.

12. $f(x) = x^{2/3}(x-3) \quad \Rightarrow \quad f'(x) = x^{2/3}(1) + (x-3)\cdot\frac{2}{3}x^{-1/3} = \frac{1}{3}x^{-1/3}\left[3x + (x-3)\cdot 2\right] = \frac{1}{3}x^{-1/3}(5x-6)$.

Interval	$x^{-1/3}$	$5x-6$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < \frac{6}{5}$	$+$	$-$	$-$	decreasing on $\left(0, \frac{6}{5}\right)$
$x > \frac{6}{5}$	$+$	$+$	$+$	increasing on $\left(\frac{6}{5}, \infty\right)$

f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = \frac{6}{5}$. Thus, $f(0) = 0$ is a local maximum value and $f\left(\frac{6}{5}\right) = \left(\frac{6}{5}\right)^{2/3}\left(-\frac{9}{5}\right) \approx -2.03$ is a local minimum value.

13. $f(x) = \dfrac{x^2-24}{x-5} \quad \Rightarrow$

$$f'(x) = \frac{(x-5)(2x) - (x^2-24)(1)}{(x-5)^2} = \frac{2x^2-10x-x^2+24}{(x-5)^2} = \frac{x^2-10x+24}{(x-5)^2} = \frac{(x-4)(x-6)}{(x-5)^2}.$$

Interval	$x-4$	$x-6$	$(x-5)^2$	$f'(x)$	f
$x < 4$	$-$	$-$	$+$	$+$	increasing on $(-\infty, 4)$
$4 < x < 5$	$+$	$-$	$+$	$-$	decreasing on $(4, 5)$
$5 < x < 6$	$+$	$-$	$+$	$-$	decreasing on $(5, 6)$
$x > 6$	$+$	$+$	$+$	$+$	increasing on $(6, \infty)$

$x = 5$ is not in the domain of f. f changes from increasing to decreasing at $x = 4$ and from decreasing to increasing at $x = 6$. Thus, $f(4) = 8$ is a local maximum value and $f(6) = 12$ is a local minimum value.

14. $f(x) = x + \dfrac{4}{x^2} = x + 4x^{-2} \quad \Rightarrow \quad f'(x) = 1 - 8x^{-3} = 1 - \dfrac{8}{x^3} = \dfrac{x^3-8}{x^3} = \dfrac{(x-2)(x^2+2x+4)}{x^3}$. The factor x^2+2x+4 is always positive and does not affect the sign of $f'(x)$.

Interval	x^3	$x-2$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

$x = 0$ is not in the domain of f. f changes from decreasing to increasing at $x = 2$. Thus, $f(2) = 3$ is a local minimum value.

15. $f(x) = \sin x + \cos x,\ 0 \le x \le 2\pi. \quad f'(x) = \cos x - \sin x = 0 \quad \Rightarrow \quad \cos x = \sin x \quad \Rightarrow \quad 1 = \dfrac{\sin x}{\cos x} \quad \Rightarrow$

$\tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4}$ or $\frac{5\pi}{4}$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad \cos x - \sin x > 0 \quad \Leftrightarrow \quad \cos x > \sin x \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{4}$ or

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$\frac{5\pi}{4} < x < 2\pi$ and $f'(x) < 0 \quad \Leftrightarrow \quad \cos x < \sin x \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{5\pi}{4}$. So f is increasing on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{5\pi}{4}, 2\pi\right)$ and f is decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

f changes from increasing to decreasing at $x = \frac{\pi}{4}$ and from decreasing to increasing at $x = \frac{5\pi}{4}$. Thus, $f\left(\frac{\pi}{4}\right) = \sqrt{2}$ is a local maximum value and $f\left(\frac{5\pi}{4}\right) = -\sqrt{2}$ is a local minimum value.

16. $f(x) = x^4 e^{-x} \quad \Rightarrow \quad f'(x) = x^4(-e^{-x}) + e^{-x}(4x^3) = x^3 e^{-x}(-x + 4)$. Thus, $f'(x) > 0$ if $0 < x < 4$ and $f'(x) < 0$ if $x < 0$ or $x > 4$. So f is increasing on $(0, 4)$ and decreasing on $(-\infty, 0)$ and $(4, \infty)$.

f changes from decreasing to increasing at $x = 0$ and from increasing to decreasing at $x = 4$. Thus, $f(0) = 0$ is a local minimum value and $f(4) = 256e^{-4}$ $[\approx 4.69]$ is a local maximum value.

17. $f(x) = x^3 - 3x^2 - 9x + 4 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x - 9 \quad \Rightarrow \quad f''(x) = 6x - 6 = 6(x - 1)$. $f''(x) > 0 \quad \Leftrightarrow \quad x > 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 1$. Thus, f is concave upward on $(1, \infty)$ and concave downward on $(-\infty, 1)$. There is an inflection point at $(1, f(1)) = (1, -7)$.

18. $f(x) = 2x^3 - 9x^2 + 12x - 3 \quad \Rightarrow \quad f'(x) = 6x^2 - 18x + 12 \quad \Rightarrow \quad f''(x) = 12x - 18 = 12\left(x - \frac{3}{2}\right)$. $f''(x) > 0 \quad \Leftrightarrow$ $x > \frac{3}{2}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < \frac{3}{2}$. Thus, f is concave upward on $\left(\frac{3}{2}, \infty\right)$ and concave downward on $\left(-\infty, \frac{3}{2}\right)$. There is an inflection point at $\left(\frac{3}{2}, \frac{3}{2}\right)$.

19. $f(x) = \sin^2 x - \cos 2x, \ 0 \le x \le \pi$. $f'(x) = 2 \sin x \cos x + 2 \sin 2x = \sin 2x + 2 \sin 2x = 3 \sin 2x$ and $f''(x) = 6 \cos 2x$. $f''(x) > 0 \quad \Leftrightarrow \quad \cos 2x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$ and $f''(x) < 0 \quad \Leftrightarrow$ $\cos 2x < 0 \quad \Leftrightarrow \quad \frac{\pi}{4} < x < \frac{3\pi}{4}$. Thus, f is concave upward on $\left(0, \frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$ and concave downward on $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$. There are inflection points at $\left(\frac{\pi}{4}, \frac{1}{2}\right)$ and $\left(\frac{3\pi}{4}, \frac{1}{2}\right)$.

20. $f(x) = \ln(2 + \sin x), \ 0 \le x \le 2\pi$. $f'(x) = \dfrac{1}{2 + \sin x}(\cos x) = \dfrac{\cos x}{2 + \sin x}$ and

$$f''(x) = \frac{(2 + \sin x)(-\sin x) - \cos x\,(\cos x)}{(2 + \sin x)^2} = \frac{-2 \sin x - \sin^2 x - \cos^2 x}{(2 + \sin x)^2} = \frac{-2 \sin x - (\sin^2 x + \cos^2 x)}{(2 + \sin x)^2}$$
$$= \frac{-2 \sin x - 1}{(2 + \sin x)^2} = -\frac{1 + 2 \sin x}{(2 + \sin x)^2}.$$

$f''(x) > 0 \quad \Rightarrow \quad 1 + 2 \sin x < 0 \quad \Leftrightarrow \quad \sin x < -\frac{1}{2} \quad \Leftrightarrow \quad \frac{7\pi}{6} < x < \frac{11\pi}{6}$ and $f''(x) < 0 \quad \Rightarrow \quad 1 + 2 \sin x > 0 \quad \Leftrightarrow$ $\sin x > -\frac{1}{2} \quad \Leftrightarrow \quad 0 < x < \frac{7\pi}{6}$ or $\frac{11\pi}{6} < x < 2\pi$. Thus, f is concave upward on $\left(\frac{7\pi}{6}, \frac{11\pi}{6}\right)$ and concave downward on $\left(0, \frac{7\pi}{6}\right)$ and $\left(\frac{11\pi}{6}, 2\pi\right)$. There are inflection points at $\left(\frac{7\pi}{6}, \ln\frac{3}{2}\right)$ and $\left(\frac{11\pi}{6}, \ln\frac{3}{2}\right)$.

21. $f(x) = \ln(x^2 + 5) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 + 5}(2x) = \dfrac{2x}{x^2 + 5}$ and

$$f''(x) = \frac{(x^2 + 5)(2) - 2x(2x)}{(x^2 + 5)^2} = \frac{2x^2 + 10 - 4x^2}{(x^2 + 5)^2} = \frac{-2x^2 + 10}{(x^2 + 5)^2} = -\frac{2(x^2 - 5)}{(x^2 + 5)^2}.$$

$f''(x) > 0 \quad \Rightarrow \quad x^2 - 5 < 0 \quad \Rightarrow \quad x^2 < 5 \quad \Rightarrow \quad -\sqrt{5} < x < \sqrt{5}$ and $f''(x) < 0 \quad \Rightarrow \quad x^2 - 5 > 0 \quad \Rightarrow \quad x^2 > 5 \quad \Rightarrow$

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$x < -\sqrt{5}$ or $x > \sqrt{5}$. Thus, f is concave upward on $\left(-\sqrt{5}, \sqrt{5}\right)$ and concave downward on $\left(-\infty, -\sqrt{5}\right)$ and $\left(\sqrt{5}, \infty\right)$. There are inflection points at $\left(-\sqrt{5}, \ln 10\right)$ and $\left(\sqrt{5}, \ln 10\right)$.

22. $f(x) = \dfrac{e^x}{e^x + 2} \quad \Rightarrow \quad f'(x) = \dfrac{(e^x + 2)e^x - e^x(e^x)}{(e^x + 2)^2} = \dfrac{e^x\left[(e^x + 2) - e^x\right]}{(e^x + 2)^2} = \dfrac{2e^x}{(e^x + 2)^2}$.

$f''(x) = \dfrac{(e^x + 2)^2 \cdot 2e^x - 2e^x \cdot 2(e^x + 2)e^x}{[(e^x + 2)^2]^2} = \dfrac{2e^x(e^x + 2)[(e^x + 2) - 2e^x]}{(e^x + 2)^4} = \dfrac{2e^x(2 - e^x)}{(e^x + 2)^3}$.

$f''(x) > 0 \quad \Leftrightarrow \quad 2 - e^x > 0 \quad \Leftrightarrow \quad e^x < 2 \quad \Leftrightarrow \quad x < \ln 2$ and $f''(x) < 0 \quad \Leftrightarrow \quad 2 - e^x < 0 \quad \Leftrightarrow \quad e^x > 2 \quad \Leftrightarrow$ $x > \ln 2$. Thus, f is concave upward on $(-\infty, \ln 2)$ and concave downward on $(\ln 2, \infty)$. There is an inflection point at $(\ln 2, f(\ln 2)) = \left(\ln 2, \dfrac{2}{2 + 2}\right) = \left(\ln 2, \frac{1}{2}\right)$.

23. (a) $f(x) = x^4 - 2x^2 + 3 \quad \Rightarrow \quad f'(x) = 4x^3 - 4x = 4x\left(x^2 - 1\right) = 4x(x + 1)(x - 1)$.

Interval	$x + 1$	$4x$	$x - 1$	$f'(x)$	f
$x < -1$	$-$	$-$	$-$	$-$	decreasing on $(-\infty, -1)$
$-1 < x < 0$	$+$	$-$	$-$	$+$	increasing on $(-1, 0)$
$0 < x < 1$	$+$	$+$	$-$	$-$	decreasing on $(0, 1)$
$x > 1$	$+$	$+$	$+$	$+$	increasing on $(1, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = -1$ and $x = 1$. Thus, $f(0) = 3$ is a local maximum value and $f(\pm 1) = 2$ are local minimum values.

(c) $f''(x) = 12x^2 - 4 = 12\left(x^2 - \frac{1}{3}\right) = 12\left(x + 1/\sqrt{3}\right)\left(x - 1/\sqrt{3}\right)$. $f''(x) > 0 \quad \Leftrightarrow \quad x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1/\sqrt{3} < x < 1/\sqrt{3}$. Thus, f is concave upward on $\left(-\infty, -\sqrt{3}/3\right)$ and $\left(\sqrt{3}/3, \infty\right)$ and concave downward on $\left(-\sqrt{3}/3, \sqrt{3}/3\right)$. There are inflection points at $\left(\pm\sqrt{3}/3, \frac{22}{9}\right)$.

24. (a) $f(x) = \dfrac{x}{x^2 + 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x^2 + 1)(1) - x(2x)}{(x^2 + 1)^2} = \dfrac{1 - x^2}{(x^2 + 1)^2} = -\dfrac{(x + 1)(x - 1)}{(x^2 + 1)^2}$. Thus, $f'(x) > 0$ if $(x + 1)(x - 1) < 0 \quad \Leftrightarrow \quad -1 < x < 1$, and $f'(x) < 0$ if $x < -1$ or $x > 1$. So f is increasing on $(-1, 1)$ and f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) f changes from decreasing to increasing at $x = -1$ and from increasing to decreasing at $x = 1$. Thus, $f(-1) = -\frac{1}{2}$ is a local minimum value and $f(1) = \frac{1}{2}$ is a local maximum value.

(c) $f''(x) = \dfrac{(x^2 + 1)^2(-2x) - (1 - x^2)[2(x^2 + 1)(2x)]}{[(x^2 + 1)^2]^2} = \dfrac{(x^2 + 1)(-2x)[(x^2 + 1) + 2(1 - x^2)]}{(x^2 + 1)^4} = \dfrac{2x(x^2 - 3)}{(x^2 + 1)^3}$.

$f''(x) > 0 \quad \Leftrightarrow \quad -\sqrt{3} < x < 0$ or $x > \sqrt{3}$, and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\sqrt{3}$ or $0 < x < \sqrt{3}$. Thus, f is concave upward on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$ and concave downward on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$. There are inflection points at $\left(-\sqrt{3}, -\sqrt{3}/4\right)$, $(0, 0)$, and $\left(\sqrt{3}, \sqrt{3}/4\right)$.

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25. (a) $f(x) = x^2 - x - \ln x \quad \Rightarrow \quad f'(x) = 2x - 1 - \dfrac{1}{x} = \dfrac{2x^2 - x - 1}{x} = \dfrac{(2x+1)(x-1)}{x}$. Thus, $f'(x) > 0$ if $x > 1$ [note that $x > 0$] and $f'(x) < 0$ if $0 < x < 1$. So f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.

(b) f changes from decreasing to increasing at $x = 1$. Thus, $f(1) = 0$ is a local minimum value.

(c) $f''(x) = 2 + 1/x^2 > 0$ for all x, so f is concave upward on $(0, \infty)$. There is no inflection point.

26. (a) $f(x) = x^2 \ln x \quad \Rightarrow \quad f'(x) = x^2(1/x) + (\ln x)(2x) = x + 2x \ln x = x(1 + 2\ln x)$. The domain of f is $(0, \infty)$, so the sign of f' is determined solely by the factor $1 + 2\ln x$. $f'(x) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$ $[\approx 0.61]$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < e^{-1/2}$. So f is increasing on $(e^{-1/2}, \infty)$ and f is decreasing on $(0, e^{-1/2})$.

(b) f changes from decreasing to increasing at $x = e^{-1/2}$. Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$ $[\approx -0.18]$ is a local minimum value.

(c) $f'(x) = x(1 + 2\ln x) \quad \Rightarrow \quad f''(x) = x(2/x) + (1 + 2\ln x) \cdot 1 = 2 + 1 + 2\ln x = 3 + 2\ln x$. $f''(x) > 0 \quad \Leftrightarrow \quad 3 + 2\ln x > 0 \quad \Leftrightarrow \quad \ln x > -3/2 \quad \Leftrightarrow \quad x > e^{-3/2}$ $[\approx 0.22]$. Thus, f is concave upward on $(e^{-3/2}, \infty)$ and f is concave downward on $(0, e^{-3/2})$. $f(e^{-3/2}) = (e^{-3/2})^2 \ln e^{-3/2} = e^{-3}(-3/2) = -3/(2e^3)$ $[\approx -0.07]$. There is a point of inflection at $\left(e^{-3/2}, f(e^{-3/2})\right) = \left(e^{-3/2}, -3/(2e^3)\right)$.

27. (a) $f(x) = xe^{2x} \quad \Rightarrow \quad f'(x) = x(2e^{2x}) + e^{2x}(1) = e^{2x}(2x + 1)$. Thus, $f'(x) > 0$ if $x > -\frac{1}{2}$ and $f'(x) < 0$ if $x < -\frac{1}{2}$. So f is increasing on $\left(-\frac{1}{2}, \infty\right)$ and f is decreasing on $\left(-\infty, -\frac{1}{2}\right)$.

(b) f changes from decreasing to increasing at $x = -\frac{1}{2}$. Thus, $f\left(-\frac{1}{2}\right) = -\frac{1}{2}e^{-1} = -1/(2e)$ $[\approx -0.18]$ is a local minimum value.

(c) $f''(x) = e^{2x}(2) + (2x + 1) \cdot 2e^{2x} = 2e^{2x}[1 + (2x + 1)] = 2e^{2x}(2x + 2) = 4e^{2x}(x + 1)$. $f''(x) > 0 \quad \Leftrightarrow \quad x > -1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -1$. Thus, f is concave upward on $(-1, \infty)$ and f is concave downward on $(-\infty, -1)$. There is an inflection point at $\left(-1, -e^{-2}\right)$, or $\left(-1, -1/e^2\right)$.

28. (a) $f(x) = \cos^2 x - 2\sin x, \quad 0 \le x \le 2\pi. \quad f'(x) = -2\cos x \sin x - 2\cos x = -2\cos x\,(1 + \sin x)$. Note that $1 + \sin x \ge 0$ [since $\sin x \ge -1$], with equality $\Leftrightarrow \quad \sin x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{2}$ [since $0 \le x \le 2\pi$] $\Rightarrow$ $\cos x = 0$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad \cos x < 0 \quad \Leftrightarrow \quad \frac{\pi}{2} < x < \frac{3\pi}{2}$ and $f'(x) < 0 \quad \Leftrightarrow \quad \cos x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and f is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) f changes from decreasing to increasing at $x = \frac{\pi}{2}$ and from increasing to decreasing at $x = \frac{3\pi}{2}$. Thus, $f\left(\frac{\pi}{2}\right) = -2$ is a local minimum value and $f\left(\frac{3\pi}{2}\right) = 2$ is a local maximum value.

(c) $f''(x) = 2\sin x\,(1 + \sin x) - 2\cos^2 x = 2\sin x + 2\sin^2 x - 2(1 - \sin^2 x)$

$$= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1)$$

so $f''(x) > 0 \quad \Leftrightarrow \quad \sin x > \frac{1}{2} \quad \Leftrightarrow \quad \frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0 \quad \Leftrightarrow \quad \sin x < \frac{1}{2}$ and $\sin x \ne -1 \quad \Leftrightarrow$

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$0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, f is concave upward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and concave downward on $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

29. $f(x) = 1 + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 6x - 6x^2 = 6x(1 - x)$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad 0 < x < 1$ and $f'(x) < 0 \quad \Rightarrow \quad x < 0$ or $x > 1$. Since f' changes from negative to positive at $x = 0$, $f(0) = 1$ is a local minimum value; and since f' changes from positive to negative at $x = 1$, $f(1) = 2$ is a local maximum value.

Second Derivative Test: $f''(x) = 6 - 12x$. $f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 1$. $f''(0) = 6 > 0 \quad \Rightarrow \quad f(0) = 1$ is a local minimum value. $f''(1) = -6 < 0 \quad \Rightarrow \quad f(1) = 2$ is a local maximum value.

Preference: For this function, the two tests are equally easy.

30. $f(x) = \dfrac{x^2}{x - 1} \quad \Rightarrow \quad f'(x) = \dfrac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.

First Derivative Test: $f'(x) > 0 \quad \Rightarrow \quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad \Rightarrow \quad 0 < x < 1$ or $1 < x < 2$. Since f' changes from positive to negative at $x = 0$, $f(0) = 0$ is a local maximum value; and since f' changes from negative to positive at $x = 2$, $f(2) = 4$ is a local minimum value.

Second Derivative Test:

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{[(x-1)^2]^2} = \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} = \frac{2}{(x-1)^3}.$$

$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 2$. $f''(0) = -2 < 0 \quad \Rightarrow \quad f(0) = 0$ is a local maximum value. $f''(2) = 2 > 0 \quad \Rightarrow \quad f(2) = 4$ is a local minimum value.

Preference: Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

31. $f'(x) = (x-4)^2(x+3)^7(x-5)^8$. The factors $(x-4)^2$ and $(x-5)^8$ are nonnegative. Hence, the sign of f' is determined by the sign of $(x+3)^7$, which is positive for $x > -3$. Thus, f increases on the intervals $(-3, 4)$, $(4, 5)$, and $(5, \infty)$. Note that f is differentiable and $f'(x) > 0$ on the interval $(-3, \infty)$ except for the numbers $x = 4$ and $x = 5$. By applying the result of Exercise 4.3.97, we can say that f is increasing on the entire interval $(-3, \infty)$.

32. (a) $f(x) = x^4(x-1)^3 \quad \Rightarrow \quad f'(x) = x^4 \cdot 3(x-1)^2 + (x-1)^3 \cdot 4x^3 = x^3(x-1)^2\,[3x + 4(x-1)] = x^3(x-1)^2(7x-4)$

The critical numbers are 0, 1, and $\frac{4}{7}$.

(b) $f''(x) = 3x^2(x-1)^2(7x-4) + x^3 \cdot 2(x-1)(7x-4) + x^3(x-1)^2 \cdot 7$

$= x^2(x-1)\,[3(x-1)(7x-4) + 2x(7x-4) + 7x(x-1)]$

Now $f''(0) = f''(1) = 0$, so the Second Derivative Test gives no information for $x = 0$ or $x = 1$.

$f''\left(\frac{4}{7}\right) = \left(\frac{4}{7}\right)^2\left(\frac{4}{7} - 1\right)\left[0 + 0 + 7\left(\frac{4}{7}\right)\left(\frac{4}{7} - 1\right)\right] = \left(\frac{4}{7}\right)^2\left(-\frac{3}{7}\right)(4)\left(-\frac{3}{7}\right) > 0$, so there is a local minimum at $x = \frac{4}{7}$.

(c) f' is positive on $(-\infty, 0)$, negative on $\left(0, \frac{4}{7}\right)$, positive on $\left(\frac{4}{7}, 1\right)$, and positive on $(1, \infty)$. So f has a local maximum at $x = 0$, a local minimum at $x = \frac{4}{7}$, and no local maximum or minimum at $x = 1$.

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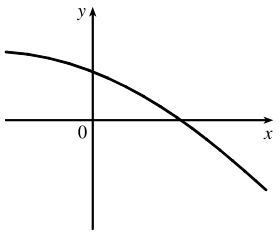
33. (a) By the Second Derivative Test, if $f'(2) = 0$ and $f''(2) = -5 < 0$, f has a local maximum at $x = 2$.

(b) If $f'(6) = 0$, we know that f has a horizontal tangent at $x = 6$. Knowing that $f''(6) = 0$ does not provide any additional information since the Second Derivative Test fails. For example, the first and second derivatives of $y = (x-6)^4$, $y = -(x-6)^4$, and $y = (x-6)^3$ all equal zero for $x = 6$, but the first has a local minimum at $x = 6$, the second has a local maximum at $x = 6$, and the third has an inflection point at $x = 6$.

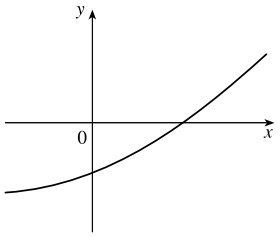
34. (a) $f'(x) < 0$ and $f''(x) < 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave downward (since the second derivative is always negative).



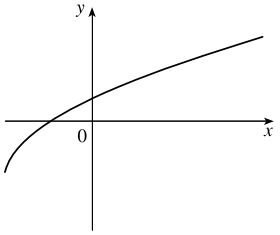
(b) $f'(x) > 0$ and $f''(x) > 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave upward (since the second derivative is always positive).



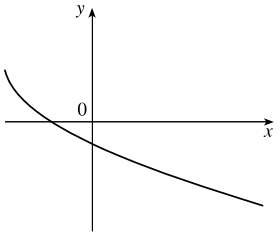
35. (a) $f'(x) > 0$ and $f''(x) < 0$ for all x

The function must be always increasing (since the first derivative is always positive) and concave downward (since the second derivative is always negative).



(b) $f'(x) < 0$ and $f''(x) > 0$ for all x

The function must be always decreasing (since the first derivative is always negative) and concave upward (since the second derivative is always positive).



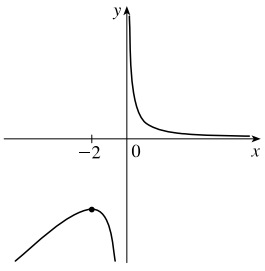
36. Vertical asymptote $x = 0$

$f'(x) > 0$ if $x < -2 \quad \Rightarrow \quad f$ is increasing on $(-\infty, -2)$.

$f'(x) < 0$ if $x > -2$ $(x \neq 0) \quad \Rightarrow \quad f$ is decreasing on $(-2, 0)$ and $(0, \infty)$.

$f''(x) < 0$ if $x < 0 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 0)$.

$f''(x) > 0$ if $x > 0 \quad \Rightarrow \quad f$ is concave upward on $(0, \infty)$.



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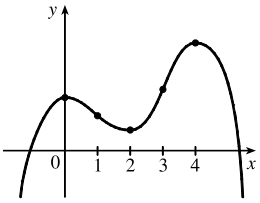
37. $f'(0) = f'(2) = f'(4) = 0 \quad \Rightarrow$ horizontal tangents at $x = 0, 2, 4$.

$f'(x) > 0$ if $x < 0$ or $2 < x < 4 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 0)$ and $(2, 4)$.

$f'(x) < 0$ if $0 < x < 2$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(0, 2)$ and $(4, \infty)$.

$f''(x) > 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave upward on $(1, 3)$.

$f''(x) < 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave downward on $(-\infty, 1)$ and $(3, \infty)$. There are inflection points when $x = 1$ and 3.



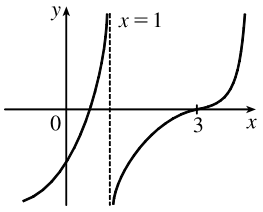
38. $f'(x) > 0$ for all $x \neq 1 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 1)$ and $(1, \infty)$.

Vertical asymptote $x = 1$

$f''(x) > 0$ if $x < 1$ or $x > 3 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 1)$ and $(3, \infty)$.

$f''(x) < 0$ if $1 < x < 3 \quad \Rightarrow \quad f$ is concave downward on $(1, 3)$.

There is an inflection point at $x = 3$.

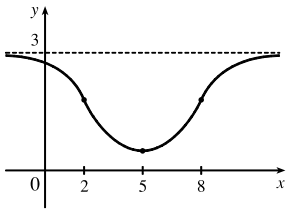


39. $f'(5) = 0 \quad \Rightarrow$ horizontal tangent at $x = 5$.

$f'(x) < 0$ when $x < 5 \quad \Rightarrow \quad f$ is decreasing on $(-\infty, 5)$.

$f'(x) > 0$ when $x > 5 \quad \Rightarrow \quad f$ is increasing on $(5, \infty)$.

$f''(2) = 0, \quad f''(8) = 0, \quad f''(x) < 0$ when $x < 2$ or $x > 8$,



$f''(x) > 0$ for $2 < x < 8 \quad \Rightarrow \quad f$ is concave upward on $(2, 8)$ and concave downward on $(-\infty, 2)$ and $(8, \infty)$.

There are inflection points at $x = 2$ and $x = 8$.

$\lim_{x \to \infty} f(x) = 3, \quad \lim_{x \to -\infty} f(x) = 3 \quad \Rightarrow \quad y = 3$ is a horizontal asymptote.

40. $f'(0) = f'(4) = 0 \quad \Rightarrow$ horizontal tangents at $x = 0$ and 4.

$f'(x) = 1$ if $x < -1 \quad \Rightarrow \quad f$ is a line with slope 1 on $(-\infty, -1)$.

$f'(x) > 0$ if $0 < x < 2 \quad \Rightarrow \quad f$ is increasing on $(0, 2)$.

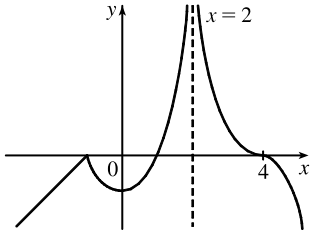
$f'(x) < 0$ if $-1 < x < 0$ or $2 < x < 4$ or $x > 4 \quad \Rightarrow \quad f$ is decreasing on $(-1, 0)$, $(2, 4)$, and $(4, \infty)$.

$\lim_{x \to 2^-} f'(x) = \infty \quad \Rightarrow \quad f'$ increases without bound as $x \to 2^-$.

$\lim_{x \to 2^+} f'(x) = -\infty \quad \Rightarrow \quad f'$ decreases without bound as $x \to 2^+$.

$f''(x) > 0$ if $-1 < x < 2$ or $2 < x < 4 \quad \Rightarrow \quad f$ is concave upward on $(-1, 2)$ and $(2, 4)$.

$f''(x) < 0$ if $x > 4 \quad \Rightarrow \quad f$ is concave downward on $(4, \infty)$.



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41. $f'(x) > 0$ if $x \neq 2 \quad \Rightarrow \quad f$ is increasing on $(-\infty, 2)$ and $(2, \infty)$.

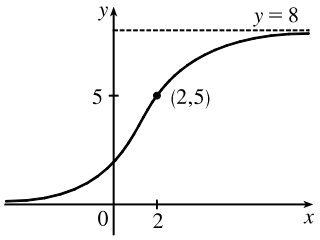
$f''(x) > 0$ if $x < 2 \quad \Rightarrow \quad f$ is concave upward on $(-\infty, 2)$.

$f''(x) < 0$ if $x > 2 \quad \Rightarrow \quad f$ is concave downward on $(2, \infty)$.

f has inflection point $(2, 5) \quad \Rightarrow \quad f$ changes concavity at the point $(2, 5)$.

$\lim_{x \to \infty} f(x) = 8 \quad \Rightarrow \quad f$ has a horizontal asymptote of $y = 8$ as $x \to \infty$.

$\lim_{x \to -\infty} f(x) = 0 \quad \Rightarrow \quad f$ has a horizontal asymptote of $y = 0$ as $x \to -\infty$.



42. (a) $\dfrac{dy}{dx} > 0$ (f is increasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point B.

(b) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} < 0$ (f is concave downward) at point E.

(c) $\dfrac{dy}{dx} < 0$ (f is decreasing) and $\dfrac{d^2y}{dx^2} > 0$ (f is concave upward) at point A.

Note: At C, $\dfrac{dy}{dx} > 0$ and $\dfrac{d^2y}{dx^2} < 0$. At D, $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} \leq 0$.

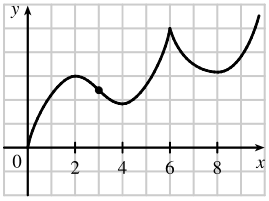
43. (a) f is increasing where f' is positive, that is, on $(0, 2)$, $(4, 6)$, and $(8, \infty)$; and decreasing where f' is negative, that is, on $(2, 4)$ and $(6, 8)$.

(b) f has local maxima where f' changes from positive to negative, at $x = 2$ and at $x = 6$, and local minima where f' changes from negative to positive, at $x = 4$ and at $x = 8$.

(c) f is concave upward (CU) where f' is increasing, that is, on $(3, 6)$ and $(6, \infty)$, and concave downward (CD) where f' is decreasing, that is, on $(0, 3)$.

(d) There is a point of inflection where f changes from being CD to being CU, that is, at $x = 3$.

(e)



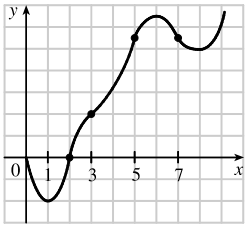
44. (a) f is increasing where f' is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where f' is negative, on $(0, 1)$ and $(6, 8)$.

(b) f has a local maximum where f' changes from positive to negative, at $x = 6$, and local minima where f' changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) f is concave upward where f' is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where f' is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where f changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)



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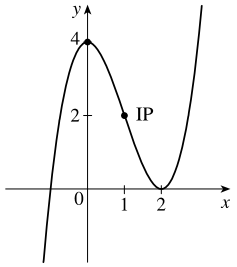
45. (a) $f(x) = x^3 - 3x^2 + 4 \quad \Rightarrow \quad f'(x) = 3x^2 - 6x = 3x(x-2)$.

Interval	$3x$	$x-2$	$f'(x)$	f
$x < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < x < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$x > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 4$ is a local maximum value and $f(2) = 0$ is a local minimum value.

(c) $f''(x) = 6x - 6 = 6(x-1)$. $f''(x) = 0 \quad \Leftrightarrow \quad x = 1$. $f''(x) > 0$ on $(1, \infty)$ and $f''(x) < 0$ on $(-\infty, 1)$. So f is concave upward on $(1, \infty)$ and f is concave downward on $(-\infty, 1)$. There is an inflection point at $(1, 2)$.

(d)

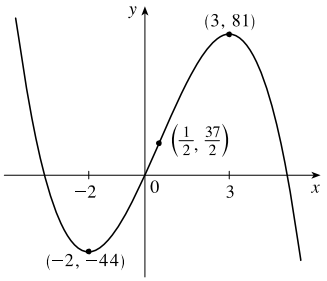


46. (a) $f(x) = 36x + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x+2)(x-3)$. $f'(x) > 0 \quad \Leftrightarrow$ $-2 < x < 3$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $x > 3$. So f is increasing on $(-2, 3)$ and f is decreasing on $(-\infty, -2)$ and $(3, \infty)$.

(b) f changes from increasing to decreasing at $x = 3$, so $f(3) = 81$ is a local maximum value. f changes from decreasing to increasing at $x = -2$, so $f(-2) = -44$ is a local minimum value.

(c) $f''(x) = 6 - 12x$. $f''(x) = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}$. $f''(x) > 0$ on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x) < 0$ on $\left(\frac{1}{2}, \infty\right)$. So f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, \frac{37}{2}\right)$.

(d)



47. (a) $f(x) = \frac{1}{2}x^4 - 4x^2 + 3 \quad \Rightarrow \quad f'(x) = 2x^3 - 8x = 2x(x^2 - 4) = 2x(x+2)(x-2)$. $f'(x) > 0 \quad \Leftrightarrow \quad -2 < x < 0$ or $x > 2$, and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $0 < x < 2$. So f is increasing on $(-2, 0)$ and $(2, \infty)$ and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

(b) f changes from increasing to decreasing at $x = 0$, so $f(0) = 3$ is a local maximum value.

f changes from decreasing to increasing at $x = \pm 2$, so $f(\pm 2) = -5$ is a local minimum value.

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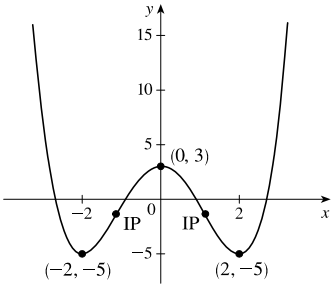
(c) $f''(x) = 6x^2 - 8 = 6\left(x^2 - \frac{4}{3}\right) = 6\left(x + \frac{2}{\sqrt{3}}\right)\left(x - \frac{2}{\sqrt{3}}\right)$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \pm\frac{2}{\sqrt{3}}$. $f''(x) > 0$ on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$ and $f''(x) < 0$ on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. So f is CU on $\left(-\infty, -\frac{2}{\sqrt{3}}\right)$ and $\left(\frac{2}{\sqrt{3}}, \infty\right)$, and f is CD on $\left(-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. There are inflection points at $\left(\pm\frac{2}{\sqrt{3}}, -\frac{13}{9}\right)$.

(d)



48. (a) $g(x) = 200 + 8x^3 + x^4 \quad \Rightarrow \quad g'(x) = 24x^2 + 4x^3 = 4x^2(6 + x) = 0$ when $x = -6$ and when $x = 0$.

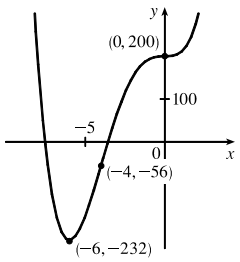
$g'(x) > 0 \quad \Leftrightarrow \quad x > -6 \quad [x \neq 0]$ and $g'(x) < 0 \quad \Leftrightarrow \quad x < -6$, so g is decreasing on $(-\infty, -6)$ and g is increasing on $(-6, \infty)$, with a horizontal tangent at $x = 0$.

(b) $g(-6) = -232$ is a local minimum value. There is no local maximum value.

(c) $g''(x) = 48x + 12x^2 = 12x(4 + x) = 0$ when $x = -4$ and when $x = 0$.

$g''(x) > 0 \quad \Leftrightarrow \quad x < -4$ or $x > 0$ and $g''(x) < 0 \quad \Leftrightarrow \quad -4 < x < 0$, so g is CU on $(-\infty, -4)$ and $(0, \infty)$, and g is CD on $(-4, 0)$. There are inflection points at $(-4, -56)$ and $(0, 200)$.

(d)



49. (a) $g(t) = 3t^4 - 8t^3 + 12 \quad \Rightarrow \quad g'(t) = 12t^3 - 24t^2 = 12t^2(t - 2)$.

Interval	$12t^2$	$t - 2$	$g'(t)$	g
$t < 0$	$+$	$-$	$-$	decreasing on $(-\infty, 0)$
$0 < t < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$t > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

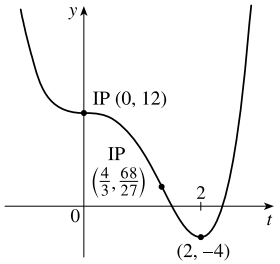
(b) g changes from decreasing to increasing at $x = 2$. Thus, $g(2) = -4$ is a local minimum value.

(c) $g''(t) = 36t^2 - 48t = 12t(3t - 4)$. $g''(t) = 0 \quad \Leftrightarrow \quad t = 0$ or $t = \frac{4}{3}$.

Interval	$12t$	$3t - 4$	$g''(t)$	g
$t < 0$	$-$	$-$	$+$	concave up on $(-\infty, 0)$
$0 < t < \frac{4}{3}$	$+$	$-$	$-$	concave down on $\left(0, \frac{4}{3}\right)$
$t > \frac{4}{3}$	$+$	$+$	$+$	concave up on $\left(\frac{4}{3}, \infty\right)$

There are inflection points at $(0, 12)$ and $\left(\frac{4}{3}, \frac{68}{27}\right)$.

(d)



50. (a) $h(x) = 5x^3 - 3x^5 \quad \Rightarrow \quad h'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1 + x)(1 - x)$. $h'(x) > 0 \quad \Leftrightarrow$ $-1 < x < 0$ and $0 < x < 1$ [note that $h'(0) = 0$] and $h'(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$. So h is increasing on $(-1, 1)$ and h is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) h changes from decreasing to increasing at $x = -1$, so $h(-1) = -2$ is a local minimum value. h changes from increasing to decreasing at $x = 1$, so $h(1) = 2$ is a local maximum value.

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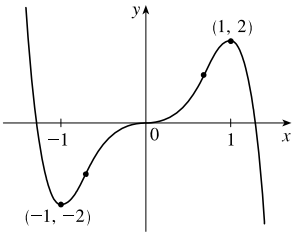
(c) $h''(x) = 30x - 60x^3 = 30x(1 - 2x^2)$. $h''(x) = 0 \Leftrightarrow x = 0$ or $1 - 2x^2 = 0 \Leftrightarrow x = 0$ or $x = \pm 1/\sqrt{2}$. $h''(x) > 0$ on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and $h''(x) < 0$ on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. So h is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and h is CD on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. There are inflection points at $\left(-1/\sqrt{2}, -7/\left(4\sqrt{2}\right)\right)$, $(0, 0)$, and $\left(1/\sqrt{2}, 7/\left(4\sqrt{2}\right)\right)$.

(d)



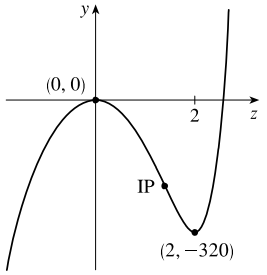
51. (a) $f(z) = z^7 - 112z^2 \Rightarrow f'(z) = 7z^6 - 224z = 7z(z^5 - 32)$. $f'(z) = 0 \Rightarrow z = 0, 2$.

Interval	$7z$	$z^5 - 32$	$f'(z)$	f
$z < 0$	$-$	$-$	$+$	increasing on $(-\infty, 0)$
$0 < z < 2$	$+$	$-$	$-$	decreasing on $(0, 2)$
$z > 2$	$+$	$+$	$+$	increasing on $(2, \infty)$

(b) f changes from increasing to decreasing at $x = 0$ and from decreasing to increasing at $x = 2$. Thus, $f(0) = 0$ is a local maximum value and $f(2) = -320$ is a local minimum value.

(c) $f''(z) = 42z^5 - 224 = 14\left(3z^5 - 16\right)$. $f''(z) = 0 \Leftrightarrow 3z^5 = 16 \Leftrightarrow z^5 = \frac{16}{3} \Leftrightarrow z = \sqrt[5]{\frac{16}{3}}$ [call this value a]. $f''(z) > 0 \Leftrightarrow z > a$ and $f''(z) < 0 \Leftrightarrow z < a$. So, f is concave up on (a, ∞) and concave down on $(-\infty, a)$. There is an inflection point at $(a, f(a)) = \left(\sqrt[5]{\frac{16}{3}}, -\frac{320}{3}\sqrt[5]{\frac{256}{9}}\right) \approx (1.398, -208.4)$.

(d)



52. (a) $f(x) = \left(x^2 - 4\right)^3 \Rightarrow f'(x) = 3(x^2 - 4)^2(2x) = 6x(x^2 - 4)^2$. Since $(x^2 - 4)^2$ is nonnegative, the sign of $f'(x)$ is determined by the sign of $6x$. Thus, $f'(x) < 0 \Leftrightarrow x < 0$ $[x \neq -2]$ and $f'(x) > 0 \Leftrightarrow x > 0$ $[x \neq 2]$. So f is increasing on $(0, 2)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(-2, 0)$. By Exercise 4.3.97, we can say that f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$. Thus, $f(0) = -64$ is a local minimum value.

(c) $f''(x) = 6x \cdot 2(x^2 - 4)(2x) + (x^2 - 4)^2 \cdot 6 = 6(x^2 - 4)[4x^2 + (x^2 - 4)]$
$= 6(x^2 - 4)(5x^2 - 4) = 6(x + 2)(x - 2)\left(\sqrt{5}\,x + 2\right)\left(\sqrt{5}\,x - 2\right)$

Interval	$x + 2$	$\sqrt{5}\,x + 2$	$\sqrt{5}\,x - 2$	$x - 2$	$f''(x)$	f
$x < -2$	$-$	$-$	$-$	$-$	$+$	concave up on $(-\infty, -2)$
$-2 < x < -\frac{2}{\sqrt{5}}$	$+$	$-$	$-$	$-$	$-$	concave down on $\left(-2, -\frac{2}{\sqrt{5}}\right)$
$-\frac{2}{\sqrt{5}} < x < \frac{2}{\sqrt{5}}$	$+$	$+$	$-$	$-$	$+$	concave up on $\left(-\frac{2}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$
$\frac{2}{\sqrt{5}} < x < 2$	$+$	$+$	$+$	$-$	$-$	concave down on $\left(\frac{2}{\sqrt{5}}, 2\right)$
$x > 2$	$+$	$+$	$+$	$+$	$+$	concave up on $(2, \infty)$

There are inflection points at $(-2, 0)$, $\left(-\frac{2}{\sqrt{5}}, -\frac{4096}{125}\right)$, $\left(\frac{2}{\sqrt{5}}, -\frac{4096}{125}\right)$, and $(2, 0)$.

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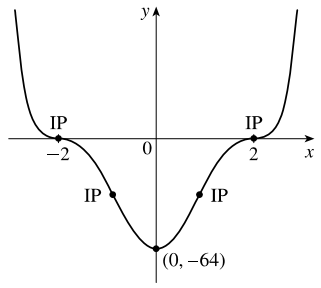
(d)



53. (a) $F(x) = x\sqrt{6-x} \;\Rightarrow$

$$F'(x) = x \cdot \tfrac{1}{2}(6-x)^{-1/2}(-1) + (6-x)^{1/2}(1) = \tfrac{1}{2}(6-x)^{-1/2}[-x + 2(6-x)] = \frac{-3x+12}{2\sqrt{6-x}}.$$

$F'(x) > 0 \;\Leftrightarrow\; -3x + 12 > 0 \;\Leftrightarrow\; x < 4$ and $F'(x) < 0 \;\Leftrightarrow\; 4 < x < 6$. So F is increasing on $(-\infty, 4)$ and F is decreasing on $(4, 6)$.

(b) F changes from increasing to decreasing at $x = 4$, so $F(4) = 4\sqrt{2}$ is a local maximum value. There is no local minimum value.

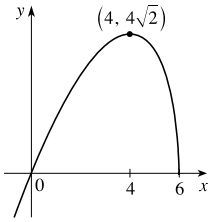
(c) $F'(x) = -\frac{3}{2}(x-4)(6-x)^{-1/2} \;\Rightarrow$

$$F''(x) = -\tfrac{3}{2}\left[(x-4)\left(-\tfrac{1}{2}(6-x)^{-3/2}(-1)\right) + (6-x)^{-1/2}(1)\right]$$

$$= -\frac{3}{2} \cdot \frac{1}{2}(6-x)^{-3/2}[(x-4) + 2(6-x)] = \frac{3(x-8)}{4(6-x)^{3/2}}$$

$F''(x) < 0$ on $(-\infty, 6)$, so F is CD on $(-\infty, 6)$. There is no inflection point.

(d)



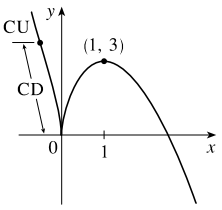
54. (a) $G(x) = 5x^{2/3} - 2x^{5/3} \;\Rightarrow\; G'(x) = \frac{10}{3}x^{-1/3} - \frac{10}{3}x^{2/3} = \frac{10}{3}x^{-1/3}(1-x) = \dfrac{10(1-x)}{3x^{1/3}}$.

$G'(x) > 0 \;\Leftrightarrow\; 0 < x < 1$ and $G'(x) < 0 \;\Leftrightarrow\; x < 0$ or $x > 1$. So G is increasing on $(0, 1)$ and G is decreasing on $(-\infty, 0)$ and $(1, \infty)$.

(b) G changes from decreasing to increasing at $x = 0$, so $G(0) = 0$ is a local minimum value. G changes from increasing to decreasing at $x = 1$, so $G(1) = 3$ is a local maximum value. Note that the First Derivative Test applies at $x = 0$ even though G' is not defined at $x = 0$, since G is continuous at 0.

(c) $G''(x) = -\frac{10}{9}x^{-4/3} - \frac{20}{9}x^{-1/3} = -\frac{10}{9}x^{-4/3}(1+2x)$. $G''(x) > 0 \;\Leftrightarrow\; x < -\frac{1}{2}$ and $G''(x) < 0 \;\Leftrightarrow\; -\frac{1}{2} < x < 0$ or $x > 0$. So G is CU on $\left(-\infty, -\frac{1}{2}\right)$ and G is CD on $\left(-\frac{1}{2}, 0\right)$ and $(0, \infty)$. The only change in concavity occurs at $x = -\frac{1}{2}$, so there is an inflection point at $\left(-\frac{1}{2}, 6/\sqrt[3]{4}\right)$.

(d)



55. (a) $C(x) = x^{1/3}(x+4) = x^{4/3} + 4x^{1/3} \;\Rightarrow\; C'(x) = \frac{4}{3}x^{1/3} + \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x+1) = \dfrac{4(x+1)}{3\sqrt[3]{x^2}}$. $C'(x) > 0$ if $-1 < x < 0$ or $x > 0$ and $C'(x) < 0$ for $x < -1$, so C is increasing on $(-1, \infty)$ and C is decreasing on $(-\infty, -1)$.

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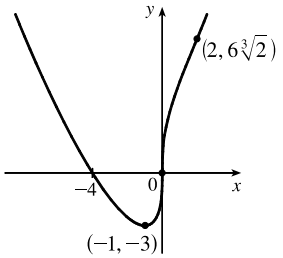
(b) $C(-1) = -3$ is a local minimum value.

(c) $C''(x) = \frac{4}{9}x^{-2/3} - \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x-2) = \dfrac{4(x-2)}{9\sqrt[3]{x^5}}$.

$C''(x) < 0$ for $0 < x < 2$ and $C''(x) > 0$ for $x < 0$ and $x > 2$, so C is concave downward on $(0, 2)$ and concave upward on $(-\infty, 0)$ and $(2, \infty)$. There are inflection points at $(0, 0)$ and $\left(2, 6\sqrt[3]{2}\right) \approx (2, 7.56)$.

(d)



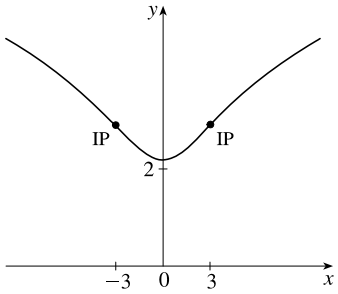
56. (a) $f(x) = \ln(x^2+9) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2+9}\cdot 2x = \dfrac{2x}{x^2+9}$. $f'(x) > 0 \;\Leftrightarrow\; 2x > 0 \;\Leftrightarrow\; x > 0$ and $f'(x) < 0 \;\Leftrightarrow\; x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x = 0$, so $f(0) = \ln 9$ is a local minimum value. There is no local maximum value.

(c) $f''(x) = \dfrac{(x^2+9)\cdot 2 - 2x(2x)}{(x^2+9)^2} = \dfrac{18-2x^2}{(x^2+9)^2} = \dfrac{-2(x+3)(x-3)}{(x^2+9)^2}$.

$f''(x) = 0 \;\Leftrightarrow\; x = \pm 3$. $f''(x) > 0$ on $(-3, 3)$ and $f''(x) < 0$ on $(-\infty, -3)$ and $(3, \infty)$. So f is CU on $(-3, 3)$, and f is CD on $(-\infty, -3)$ and $(3, \infty)$. There are inflection points at $(\pm 3, \ln 18)$.

(d)



57. (a) $f(\theta) = 2\cos\theta + \cos^2\theta,\ 0 \le \theta \le 2\pi \;\Rightarrow\; f'(\theta) = -2\sin\theta + 2\cos\theta\,(-\sin\theta) = -2\sin\theta\,(1+\cos\theta)$.

$f'(\theta) = 0 \;\Leftrightarrow\; \theta = 0, \pi$, and 2π. $f'(\theta) > 0 \;\Leftrightarrow\; \pi < \theta < 2\pi$ and $f'(\theta) < 0 \;\Leftrightarrow\; 0 < \theta < \pi$. So f is increasing on $(\pi, 2\pi)$ and f is decreasing on $(0, \pi)$.

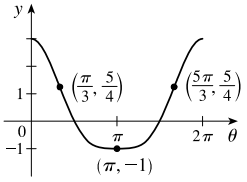
(b) $f(\pi) = -1$ is a local minimum value.

(c) $f'(\theta) = -2\sin\theta\,(1+\cos\theta) \;\Rightarrow$

$$\begin{aligned} f''(\theta) &= -2\sin\theta\,(-\sin\theta) + (1+\cos\theta)(-2\cos\theta) = 2\sin^2\theta - 2\cos\theta - 2\cos^2\theta \\ &= 2(1-\cos^2\theta) - 2\cos\theta - 2\cos^2\theta = -4\cos^2\theta - 2\cos\theta + 2 \\ &= -2(2\cos^2\theta + \cos\theta - 1) = -2(2\cos\theta - 1)(\cos\theta + 1) \end{aligned}$$

Since $-2(\cos\theta + 1) < 0$ [for $\theta \ne \pi$], $f''(\theta) > 0 \;\Rightarrow\; 2\cos\theta - 1 < 0 \;\Rightarrow\; \cos\theta < \frac{1}{2} \;\Rightarrow\; \frac{\pi}{3} < \theta < \frac{5\pi}{3}$ and $f''(\theta) < 0 \;\Rightarrow\; \cos\theta > \frac{1}{2} \;\Rightarrow\; 0 < \theta < \frac{\pi}{3}$ or $\frac{5\pi}{3} < \theta < 2\pi$. So f is CU on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$ and f is CD on $\left(0, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are points of inflection at $\left(\frac{\pi}{3}, f\left(\frac{\pi}{3}\right)\right) = \left(\frac{\pi}{3}, \frac{5}{4}\right)$ and $\left(\frac{5\pi}{3}, f\left(\frac{5\pi}{3}\right)\right) = \left(\frac{5\pi}{3}, \frac{5}{4}\right)$.

(d)



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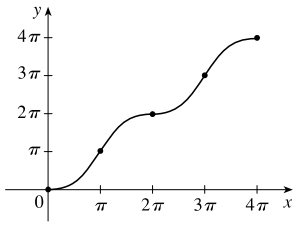
58. (a) $S(x) = x - \sin x,\ 0 \le x \le 4\pi \quad \Rightarrow \quad S'(x) = 1 - \cos x.\ S'(x) = 0 \quad \Leftrightarrow \quad \cos x = 1 \quad \Leftrightarrow \quad x = 0, 2\pi$, and 4π.
$S'(x) > 0 \quad \Leftrightarrow \quad \cos x < 1$, which is true for all x except integer multiples of 2π, so S is increasing on $(0, 4\pi)$ since $S'(2\pi) = 0$.

(b) There is no local maximum or minimum.

(c) $S''(x) = \sin x.\ S''(x) > 0$ if $0 < x < \pi$ or $2\pi < x < 3\pi$, and $S''(x) < 0$ if $\pi < x < 2\pi$ or $3\pi < x < 4\pi$. So S is CU on $(0, \pi)$ and $(2\pi, 3\pi)$, and S is CD on $(\pi, 2\pi)$ and $(3\pi, 4\pi)$. There are inflection points at (π, π), $(2\pi, 2\pi)$, and $(3\pi, 3\pi)$.

(d)



59. $f(x) = 1 + \dfrac{1}{x} - \dfrac{1}{x^2}$ has domain $(-\infty, 0) \cup (0, \infty)$.

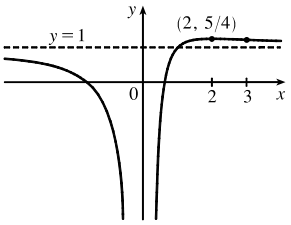
(a) $\lim\limits_{x \to \pm\infty} \left(1 + \dfrac{1}{x} - \dfrac{1}{x^2}\right) = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to 0^+} \left(1 + \dfrac{1}{x} - \dfrac{1}{x^2}\right) = \lim\limits_{x \to 0^+} \left(\dfrac{x^2 + x - 1}{x^2}\right) = -\infty$ since $(x^2 + x - 1) \to -1$ and $x^2 \to 0$ as $x \to 0^+$ [a similar argument can be made for $x \to 0^-$], so $x = 0$ is a VA.

(b) $f'(x) = -\dfrac{1}{x^2} + \dfrac{2}{x^3} = -\dfrac{1}{x^3}(x - 2).\ f'(x) = 0 \quad \Leftrightarrow \quad x = 2.\ f'(x) > 0 \quad \Leftrightarrow \quad 0 < x < 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$ or $x > 2$. So f is increasing on $(0, 2)$ and f is decreasing on $(-\infty, 0)$ and $(2, \infty)$.

(c) f changes from increasing to decreasing at $x = 2$, so $f(2) = \frac{5}{4}$ is a local maximum value. There is no local minimum value.

(d) $f''(x) = \dfrac{2}{x^3} - \dfrac{6}{x^4} = \dfrac{2}{x^4}(x - 3).\ f''(x) = 0 \quad \Leftrightarrow \quad x = 3.\ f''(x) > 0 \quad \Leftrightarrow$ $x > 3$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 0$ or $0 < x < 3$. So f is CU on $(3, \infty)$ and f is CD on $(-\infty, 0)$ and $(0, 3)$. There is an inflection point at $\left(3, \frac{11}{9}\right)$.

(e)



60. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$ has domain $\mathbb{R}$.

(a) $\lim\limits_{x \to \pm\infty} \dfrac{x^2 - 4}{x^2 + 4} = \lim\limits_{x \to \pm\infty} \dfrac{1 - 4/x^2}{1 + 4/x^2} = \dfrac{1}{1} = 1$, so $y = 1$ is a HA. There is no vertical asymptote.

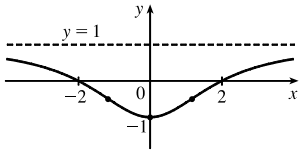
(b) $f'(x) = \dfrac{(x^2 + 4)(2x) - (x^2 - 4)(2x)}{(x^2 + 4)^2} = \dfrac{2x[(x^2 + 4) - (x^2 - 4)]}{(x^2 + 4)^2} = \dfrac{16x}{(x^2 + 4)^2}.\quad f'(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$. So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(c) f changes from decreasing to increasing at $x = 0$, so $f(0) = -1$ is a local minimum value.

(d) $f''(x) = \dfrac{(x^2 + 4)^2(16) - 16x \cdot 2(x^2 + 4)(2x)}{[(x^2 + 4)^2]^2} = \dfrac{16(x^2 + 4)[(x^2 + 4) - 4x^2]}{(x^2 + 4)^4} = \dfrac{16(4 - 3x^2)}{(x^2 + 4)^3}.$

$f''(x) = 0 \quad \Leftrightarrow \quad x = \pm 2/\sqrt{3}.\quad f''(x) > 0 \quad \Leftrightarrow \quad -2/\sqrt{3} < x < 2/\sqrt{3}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -2/\sqrt{3}$ or $x > 2/\sqrt{3}$. So f is CU on $\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$ and f is CD on $\left(-\infty, -2/\sqrt{3}\right)$ and $\left(2/\sqrt{3}, \infty\right)$. There are inflection points at $\left(\pm 2/\sqrt{3}, -\frac{1}{2}\right)$.

(e)



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61. $f(x) = e^{-2/x}$ has domain $(-\infty, 0) \cup (0, \infty)$.

(a) $\lim_{x\to\pm\infty} e^{-2/x} = \lim_{x\to\pm\infty} \dfrac{1}{e^{2/x}} = \dfrac{1}{e^0} = 1$, so $y = 1$ is a HA. $\lim_{x\to 0^-} e^{-2/x} = \infty$, so $x = 0$ is a VA.

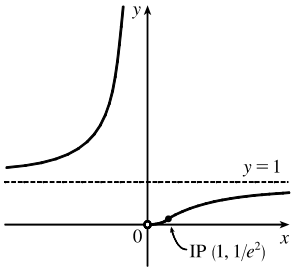
(b) $f'(x) = e^{-2/x}\left(\dfrac{2}{x^2}\right) = \dfrac{2}{x^2 e^{2/x}}$. f is always positive on its domain, so f increases on $(-\infty, 0)$ and $(0, \infty)$.

(c) f has no local maximum or minimum value.

(d) $f''(x) = \dfrac{x^2e^{2/x}\cdot 0 - 2\cdot\left[x^2\cdot e^{2/x}\left(-\dfrac{2}{x^2}\right) + e^{2/x}\cdot 2x\right]}{\left(x^2e^{2/x}\right)^2} = \dfrac{4e^{2/x}(1-x)}{\left(x^2e^{2/x}\right)^2} = \dfrac{4(1-x)}{x^4e^{2/x}}$

$f''(x) > 0 \quad\Leftrightarrow\quad x < 1$ $[x \neq 0]$ and $f''(x) < 0 \quad\Leftrightarrow\quad x > 1$ [since x^4 and $e^{2/x}$ are positive for $x \neq 0$], so f is concave up on $(-\infty, 0)$ and $(0, 1)$, and f is concave down on $(1, \infty)$. There is an inflection point at $\left(1, e^{-2}\right) \approx (1, 0.14)$.

(e)



62. $f(x) = \dfrac{e^x}{1-e^x}$ has domain $\{x \mid 1 - e^x \neq 0\} = \{x \mid e^x \neq 1\} = \{x \mid x \neq 0\}$.

(a) $\lim_{x\to\infty} \dfrac{e^x}{1-e^x} = \lim_{x\to\infty} \dfrac{e^x/e^x}{(1-e^x)/e^x} = \lim_{x\to\infty} \dfrac{1}{1/e^x - 1} = \dfrac{1}{0-1} = -1$, so $y = -1$ is a HA.

$\lim_{x\to-\infty} \dfrac{e^x}{1-e^x} = \dfrac{0}{1-0} = 0$, so $y = 0$ is a HA. $\lim_{x\to 0^+} \dfrac{e^x}{1-e^x} = -\infty$ and $\lim_{x\to 0^-} \dfrac{e^x}{1-e^x} = \infty$, so $x = 0$ is a VA.

(b) $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x)+e^x]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$.

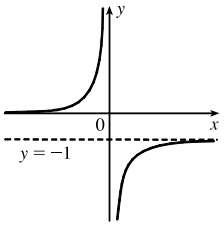
(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(1-e^x)^2e^x - e^x\cdot 2(1-e^x)(-e^x)}{[(1-e^x)^2]^2}$

$= \dfrac{(1-e^x)e^x[(1-e^x)+2e^x]}{(1-e^x)^4} = \dfrac{e^x(e^x+1)}{(1-e^x)^3}$

$f''(x) > 0 \quad\Leftrightarrow\quad (1-e^x)^3 > 0 \quad\Leftrightarrow\quad e^x < 1 \quad\Leftrightarrow\quad x < 0$ and $f''(x) < 0 \quad\Leftrightarrow\quad x > 0$. So f is CU on $(-\infty, 0)$ and f is CD on $(0, \infty)$. There is no inflection point.

(e)



63. $f(x) = e^{-x^2}$ has domain $\mathbb{R}$.

(a) $\lim_{x\to\pm\infty} e^{-x^2} = \lim_{x\to\pm\infty} \dfrac{1}{e^{x^2}} = 0$, so $y = 0$ is a HA. There is no VA.

(b) $f(x) = e^{-x^2} \quad\Rightarrow\quad f'(x) = e^{-x^2}(-2x)$. $f'(x) = 0 \quad\Leftrightarrow\quad x = 0$. $f'(x) > 0 \quad\Leftrightarrow\quad x < 0$ and $f'(x) < 0 \quad\Leftrightarrow\quad x > 0$. So f is increasing on $(-\infty, 0)$ and f is decreasing on $(0, \infty)$.

(c) f changes from increasing to decreasing at $x = 0$, so $f(0) = 1$ is a local maximum value. There is no local minimum value.

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(d) $f''(x) = e^{-x^2}(-2) + (-2x)e^{-x^2}(-2x) = -2e^{-x^2}(1-2x^2)$.

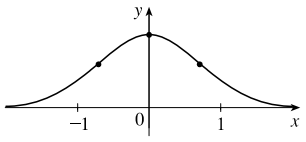
$f''(x) = 0 \quad \Leftrightarrow \quad x^2 = \frac{1}{2} \quad \Leftrightarrow \quad x = \pm 1/\sqrt{2}$. $f''(x) > 0 \quad \Leftrightarrow$

$x < -1/\sqrt{2}$ or $x > 1/\sqrt{2}$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1/\sqrt{2} < x < 1/\sqrt{2}$. So

f is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(1/\sqrt{2}, \infty\right)$, and f is CD on $\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$.

There are inflection points at $\left(\pm 1/\sqrt{2}, e^{-1/2}\right)$.

(e)



64. $f(x) = x - \frac{1}{6}x^2 - \frac{2}{3}\ln x$ has domain $(0, \infty)$.

(a) $\lim_{x \to 0^+} \left(x - \frac{1}{6}x^2 - \frac{2}{3}\ln x\right) = \infty$ since $\ln x \to -\infty$ as $x \to 0^+$, so $x = 0$ is a VA. There is no HA.

(b) $f'(x) = 1 - \frac{1}{3} - \frac{2}{3x} = \frac{3x - x^2 - 2}{3x} = \frac{-(x^2 - 3x + 2)}{3x} = \frac{-(x-1)(x-2)}{3x}$. $f'(x) > 0 \quad \Leftrightarrow$

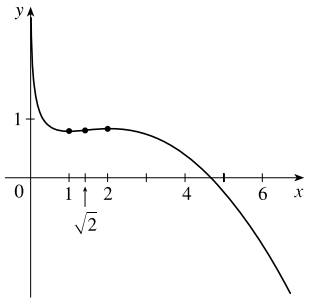
$(x-1)(x-2) < 0 \quad \Leftrightarrow \quad 1 < x < 2$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < 1$ or $x > 2$. So f is increasing on $(1, 2)$ and f is decreasing on $(0, 1)$ and $(2, \infty)$.

(c) f changes from decreasing to increasing at $x = 1$, so $f(1) = \frac{5}{6}$ is a local minimum value. f changes from increasing to decreasing at $x = 2$, so $f(2) = \frac{4}{3} - \frac{2}{3}\ln 2 \approx 0.87$ is a local maximum value.

(d) $f''(x) = -\frac{1}{3} + \frac{2}{3x^2} = \frac{2 - x^2}{3x^2}$. $f''(x) > 0 \quad \Leftrightarrow \quad 0 < x < \sqrt{2}$ and

$f''(x) < 0 \quad \Leftrightarrow \quad x > \sqrt{2}$. So f is CU on $\left(0, \sqrt{2}\right)$ and f is CD on $\left(\sqrt{2}, \infty\right)$. There is an inflection point at $\left(\sqrt{2}, \sqrt{2} - \frac{1}{3} - \frac{1}{3}\ln 2\right)$.

(e)



65. $f(x) = \ln(1 - \ln x)$ is defined when $x > 0$ (so that $\ln x$ is defined) and $1 - \ln x > 0$ [so that $\ln(1 - \ln x)$ is defined]. The second condition is equivalent to $1 > \ln x \quad \Leftrightarrow \quad x < e$, so f has domain $(0, e)$.

(a) As $x \to 0^+$, $\ln x \to -\infty$, so $1 - \ln x \to \infty$ and $f(x) \to \infty$. As $x \to e^-$, $\ln x \to 1^-$, so $1 - \ln x \to 0^+$ and $f(x) \to -\infty$. Thus, $x = 0$ and $x = e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x) = \dfrac{1}{1 - \ln x}\left(-\dfrac{1}{x}\right) = -\dfrac{1}{x(1 - \ln x)} < 0$ on $(0, e)$. Thus, f is decreasing on its domain, $(0, e)$.

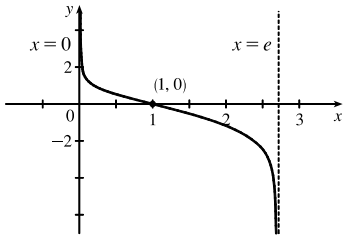
(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $f''(x) = -\dfrac{-[x(1 - \ln x)]'}{[x(1 - \ln x)]^2} = \dfrac{x(-1/x) + (1 - \ln x)}{x^2(1 - \ln x)^2}$

$= -\dfrac{\ln x}{x^2(1 - \ln x)^2}$

so $f''(x) > 0 \quad \Leftrightarrow \quad \ln x < 0 \quad \Leftrightarrow \quad 0 < x < 1$. Thus, f is CU on $(0, 1)$ and CD on $(1, e)$. There is an inflection point at $(1, 0)$.

(e)



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66. $f(x) = e^{\arctan x}$ has domain $\mathbb{R}$.

(a) $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$, so $\lim_{x\to\infty} e^{\arctan x} = e^{\pi/2}$ $[\approx 4.81]$, so $y = e^{\pi/2}$ is a HA.

$\lim_{x\to-\infty} e^{\arctan x} = e^{-\pi/2}$ $[\approx 0.21]$, so $y = e^{-\pi/2}$ is a HA. No VA.

(b) $f(x) = e^{\arctan x} \quad \Rightarrow \quad f'(x) = e^{\arctan x} \cdot \dfrac{1}{1+x^2} > 0$ for all x. Thus, f is increasing on $\mathbb{R}$.

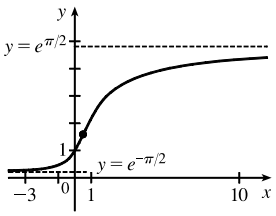
(c) There is no local maximum or minimum.

(d) $f''(x) = e^{\arctan x}\left[\dfrac{-2x}{(1+x^2)^2}\right] + \dfrac{1}{1+x^2} \cdot e^{\arctan x} \cdot \dfrac{1}{1+x^2}$

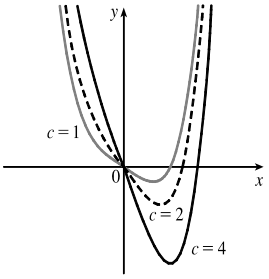
$= \dfrac{e^{\arctan x}}{(1+x^2)^2}(-2x+1)$

$f''(x) > 0 \quad \Leftrightarrow \quad -2x+1 > 0 \quad \Leftrightarrow \quad x < \frac{1}{2}$ and $f''(x) < 0 \quad \Leftrightarrow$ $x > \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, e^{\arctan(1/2)}\right) \approx \left(\frac{1}{2}, 1.59\right)$.

(e)



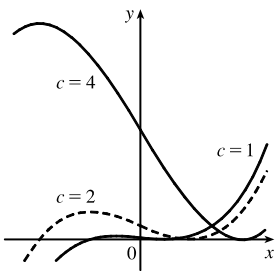
67. $f(x) = x^4 - cx, c > 0 \quad \Rightarrow \quad f'(x) = 4x^3 - c \quad \Rightarrow \quad f'(x) = 0 \quad \Leftrightarrow \quad x = \sqrt[3]{c/4}$. $f'(x) > 0 \quad \Leftrightarrow \quad x > \sqrt[3]{c/4}$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < \sqrt[3]{c/4}$. Thus, f is increasing on $\left(\sqrt[3]{c/4}, \infty\right)$ and decreasing on $\left(-\infty, \sqrt[3]{c/4}\right)$. f changes from decreasing to increasing at $x = \sqrt[3]{c/4}$. Thus, $f\left(\sqrt[3]{c/4}\right)$ is a local minimum value. $f''(x) = 12x^2$ is positive except at $x = 0$, so f is concave up on $(-\infty, 0)$ and $(0, \infty)$. There are no inflection points.



The members of this family have one local minimum point with increasing x-coordinate and decreasing y-coordinate as c increases. The graphs are concave up on $(-\infty, 0)$ and $(0, \infty)$. Since the graphs are continuous at $x = 0$, and the graphs lie above their tangents, we can say that the graphs are concave up on $(-\infty, \infty)$. There is no inflection point.

68. $f(x) = x^3 - 3c^2x + 2c^3, c > 0 \quad \Rightarrow \quad f'(x) = 3x^2 - 3c^2 = 3(x^2 - c^2) \quad \Rightarrow \quad f'(x) > 0 \quad \Leftrightarrow \quad |x| > c$ and $f'(x) < 0 \quad \Leftrightarrow \quad |x| < c$. Thus, f is increasing on $(-\infty, -c)$ and (c, ∞), and f is decreasing on $(-c, c)$. f changes from increasing to decreasing at $x = -c$ and from decreasing to increasing at $x = c$. Thus, $f(-c) = 4c^3$ is a local maximum value and $f(c) = 0$ is a local minimum value.

$f''(x) = 6x$, so $f''(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < 0$. Thus, f is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. There is an inflection point at $\left(0, 2c^3\right)$.



The members of this family have a local maximum point that moves higher and to the left as c increases, and there is a local minimum point on the x-axis that moves to the right as c increases. The graphs are concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. There is an inflection point on the y-axis that moves higher as c increases.

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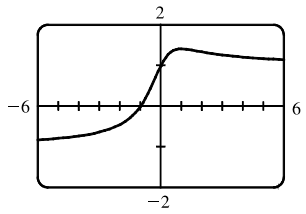
69. (a)



From the graph, we get an estimate of $f(1) \approx 1.41$ as a local maximum value, and no local minimum value.

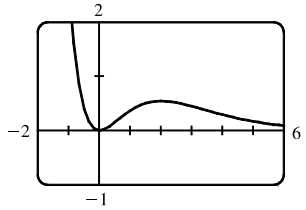
$f(x) = \dfrac{x+1}{\sqrt{x^2+1}} \quad \Rightarrow \quad f'(x) = \dfrac{1-x}{(x^2+1)^{3/2}}.$

$f'(x) = 0 \quad \Leftrightarrow \quad x = 1.\ f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ is the exact value.

(b) From the graph in part (a), f increases most rapidly somewhere between $x = -\frac{1}{2}$ and $x = -\frac{1}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'.

$f''(x) = \dfrac{2x^2 - 3x - 1}{(x^2+1)^{5/2}} = 0 \quad \Leftrightarrow \quad x = \dfrac{3 \pm \sqrt{17}}{4}$. By the First Derivative Test applied to f', $x = \dfrac{3+\sqrt{17}}{4}$ corresponds to the *minimum* value of f' and the maximum value of f' occurs at $x = \frac{3-\sqrt{17}}{4} \approx -0.28$.

70. (a)



Tracing the graph gives us estimates of $f(0) = 0$ for a local minimum value and $f(2) = 0.54$ for a local maximum value.

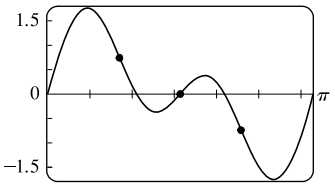
$f(x) = x^2 e^{-x} \quad \Rightarrow \quad f'(x) = xe^{-x}(2-x). \quad f'(x) = 0 \quad \Leftrightarrow \quad x = 0$ or 2.

$f(0) = 0$ and $f(2) = 4e^{-2}$ are the exact values.

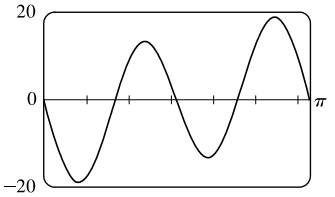
(b) From the graph in part (a), f increases most rapidly around $x = \frac{3}{4}$. To find the exact value, we need to find the maximum value of f', which we can do by finding the critical numbers of f'. $f''(x) = e^{-x}\left(x^2 - 4x + 2\right) = 0 \quad \Rightarrow$ $x = 2 \pm \sqrt{2}$. By the First Derivative Test applied to f', $x = 2 + \sqrt{2}$ corresponds to the *minimum* value of f' and the maximum value of f' is at $\left(2 - \sqrt{2}, \left(2 - \sqrt{2}\right)^2 e^{-2+\sqrt{2}}\right) \approx (0.59, 0.19)$.

71. $f(x) = \sin 2x + \sin 4x \quad \Rightarrow \quad f'(x) = 2\cos 2x + 4\cos 4x \quad \Rightarrow \quad f''(x) = -4\sin 2x - 16\sin 4x$

(a) From the graph of f, it seems that f is CD on $(0, 0.8)$, CU on $(0.8, 1.6)$, CD on $(1.6, 2.3)$, and CU on $(2.3, \pi)$. The inflection points appear to be at $(0.8, 0.7)$, $(1.6, 0)$, and $(2.3, -0.7)$.



(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(0, 0.85)$, CU on $(0.85, 1.57)$, CD on $(1.57, 2.29)$, and CU on $(2.29, \pi)$. Refined estimates of the inflection points are $(0.85, 0.74)$, $(1.57, 0)$, and $(2.29, -0.74)$.



72. $f(x) = (x-1)^2(x+1)^3 \quad \Rightarrow$

$f'(x) = (x-1)^2 3(x+1)^2 + (x+1)^3 2(x-1)$

$= (x-1)(x+1)^2\left[3(x-1) + 2(x+1)\right] = (x-1)(x+1)^2(5x-1) \quad \Rightarrow$

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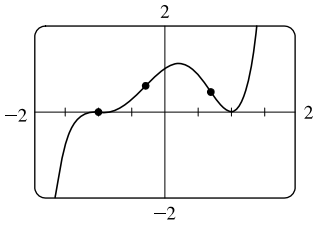
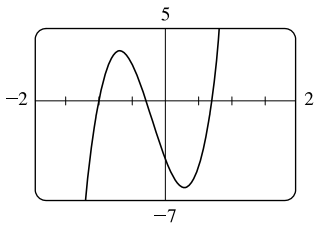
$$\begin{aligned} f''(x) &= (1)(x+1)^2(5x-1) + (x-1)(2)(x+1)(5x-1) + (x-1)(x+1)^2(5) \\ &= (x+1)[(x+1)(5x-1) + 2(x-1)(5x-1) + 5(x-1)(x+1)] \\ &= (x+1)[5x^2 + 4x - 1 + 10x^2 - 12x + 2 + 5x^2 - 5] \\ &= (x+1)(20x^2 - 8x - 4) = 4(x+1)(5x^2 - 2x - 1) \end{aligned}$$

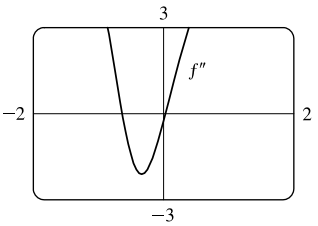
(a) From the graph of f, it seems that f is CD on $(-\infty, -1)$, CU on $(-1, -0.3)$, CD on $(-0.3, 0.7)$, and CU on $(0.7, \infty)$. The inflection points appear to be at $(-1, 0)$, $(-0.3, 0.6)$, and $(0.7, 0.5)$.



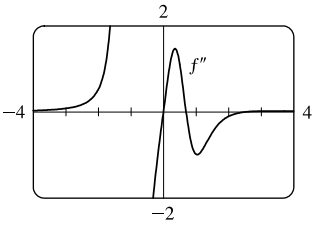
(b) From the graph of f'' (and zooming in near the zeros), it seems that f is CD on $(-1, 0)$, CU on $(-1, -0.29)$, CD on $(-0.29, 0.69)$, and CU on $(0.69, \infty)$. Refined estimates of the inflection points are $(-1, 0)$, $(-0.29, 0.60)$, and $(0.69, 0.46)$.



73. $f(x) = \dfrac{x^4 + x^3 + 1}{\sqrt{x^2 + x + 1}}$. In Maple, we define f and then use the command `plot(diff(diff(f,x),x),x=-2..2);`. In Mathematica, we define f and then use `Plot[Dt[Dt[f,x],x],{x,-2,2}]`. We see that $f'' > 0$ for $x < -0.6$ and $x > 0.0$ $[\approx 0.03]$ and $f'' < 0$ for $-0.6 < x < 0.0$. So f is CU on $(-\infty, -0.6)$ and $(0.0, \infty)$ and CD on $(-0.6, 0.0)$.



74. $f(x) = \dfrac{x^2 \tan^{-1} x}{1 + x^3}$. It appears that f'' is positive (and thus f is concave upward) on $(-\infty, -1)$, $(0, 0.7)$, and $(2.5, \infty)$; and f'' is negative (and thus f is concave downward) on $(-1, 0)$ and $(0.7, 2.5)$.



75. (a) The rate of increase of the population is initially very small, then gets larger until it reaches a maximum at about $t = 8$ hours, and decreases toward 0 as the population begins to level off.

(b) The rate of increase has its maximum value at $t = 8$ hours.

(c) The population function is concave upward on $(0, 8)$ and concave downward on $(8, 18)$.

(d) At $t = 8$, the population is about 350, so the inflection point is about $(8, 350)$.

76. If $S(t)$ is the average SAT score as a function of time t, then $S'(t) < 0$ (since the SAT scores are declining) and $S''(t) > 0$ (since the rate of decrease of the scores is increasing—becoming less negative).

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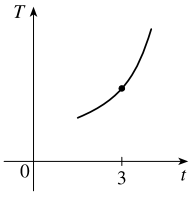
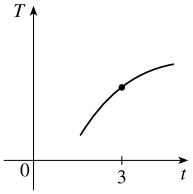
77. If $D(t)$ is the size of the national deficit as a function of time t, then at the time of the speech $D'(t) > 0$ (since the deficit is increasing), and $D''(t) < 0$ (since the rate of increase of the deficit is decreasing).

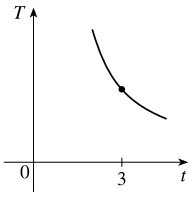
78. (a) I'm very unhappy. It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, and $f''(3) = 4$ indicates that the rate of increase is increasing. (The temperature is rapidly getting warmer.)



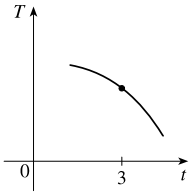
(b) I'm still unhappy, but not as unhappy as in part (a). It's uncomfortably hot and $f'(3) = 2$ indicates that the temperature is increasing, but $f''(3) = -4$ indicates that the rate of increase is decreasing. (The temperature is slowly getting warmer.)



(c) I'm somewhat happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, but $f''(3) = 4$ indicates that the rate of change is increasing. (The rate of change is negative but it's becoming less negative. The temperature is slowly getting cooler.)

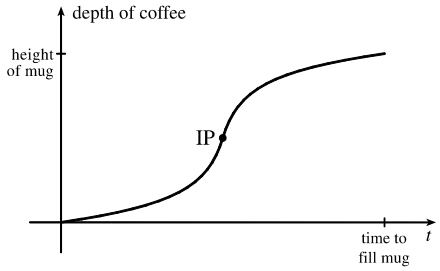


(d) I'm very happy. It's uncomfortably hot and $f'(3) = -2$ indicates that the temperature is decreasing, and $f''(3) = -4$ indicates that the rate of change is decreasing, that is, becoming more negative. (The temperature is rapidly getting cooler.)



79. Most students learn more in the third hour of studying than in the eighth hour, so $K(3) - K(2)$ is larger than $K(8) - K(7)$. In other words, as you begin studying for a test, the rate of knowledge gain is large and then starts to taper off, so $K'(t)$ decreases and the graph of K is concave downward.

80. At first the depth increases slowly because the base of the mug is wide. But as the mug narrows, the coffee rises more quickly. Thus, the depth d increases at an increasing rate and its graph is concave upward. The rate of increase of d has a maximum where the mug is narrowest; that is, when the mug is half full. It is there that the inflection point (IP) occurs. Then the rate of increase of d starts to decrease as the mug widens and the graph becomes concave down.



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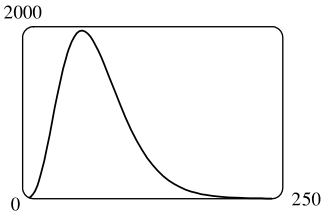
81. $S(t) = At^p e^{-kt}$ with $A = 0.01$, $p = 4$, and $k = 0.07$. We will find the zeros of f'' for $f(t) = t^p e^{-kt}$.



$f'(t) = t^p(-ke^{-kt}) + e^{-kt}(pt^{p-1}) = e^{-kt}(-kt^p + pt^{p-1})$

$f''(t) = e^{-kt}(-kpt^{p-1} + p(p-1)t^{p-2}) + (-kt^p + pt^{p-1})(-ke^{-kt})$

$= t^{p-2}e^{-kt}[-kpt + p(p-1) + k^2t^2 - kpt]$

$= t^{p-2}e^{-kt}(k^2t^2 - 2kpt + p^2 - p)$

Using the given values of p and k gives us $f''(t) = t^2e^{-0.07t}(0.0049t^2 - 0.56t + 12)$. So $S''(t) = 0.01f''(t)$ and its zeros are $t = 0$ and the solutions of $0.0049t^2 - 0.56t + 12 = 0$, which are $t_1 = \frac{200}{7} \approx 28.57$ and $t_2 = \frac{600}{7} \approx 85.71$.

At t_1 minutes, the rate of increase of the level of medication in the bloodstream is at its greatest and at t_2 minutes, the rate of decrease is the greatest.

82. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \quad \Rightarrow \quad f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$.

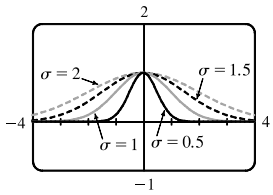
$f'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find $f''(x) = -\dfrac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \dfrac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2)$.

$f''(x) = 0 \quad \Leftrightarrow \quad x^2 = \sigma^2 \quad \Leftrightarrow \quad x = \pm\sigma$. $\quad f''(x) < 0 \quad \Leftrightarrow \quad x^2 < \sigma^2 \quad \Leftrightarrow \quad -\sigma < x < \sigma$.

So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $(\pm\sigma, e^{-1/2})$.

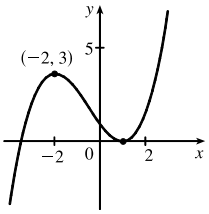
(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

83. $f(x) = ax^3 + bx^2 + cx + d \quad \Rightarrow \quad f'(x) = 3ax^2 + 2bx + c$.



We are given that $f(1) = 0$ and $f(-2) = 3$, so $f(1) = a + b + c + d = 0$ and $f(-2) = -8a + 4b - 2c + d = 3$. Also $f'(1) = 3a + 2b + c = 0$ and $f'(-2) = 12a - 4b + c = 0$ by Fermat's Theorem. Solving these four equations, we get $a = \frac{2}{9}$, $b = \frac{1}{3}$, $c = -\frac{4}{3}$, $d = \frac{7}{9}$, so the function is $f(x) = \frac{1}{9}\left(2x^3 + 3x^2 - 12x + 7\right)$.

84. $f(x) = axe^{bx^2} \quad \Rightarrow \quad f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}(2bx^2 + 1)$. For $f(2) = 1$ to be a maximum value, we must have $f'(2) = 0$. $f(2) = 1 \quad \Rightarrow \quad 1 = 2ae^{4b}$ and $f'(2) = 0 \quad \Rightarrow \quad 0 = (8b + 1)ae^{4b}$. So $8b + 1 = 0 \quad [a \neq 0] \quad \Rightarrow$ $b = -\frac{1}{8}$ and now $1 = 2ae^{-1/2} \quad \Rightarrow \quad a = \sqrt{e}/2$.

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85. $y=\dfrac{1+x}{1+x^2} \;\Rightarrow\; y'=\dfrac{(1+x^2)(1)-(1+x)(2x)}{(1+x^2)^2}=\dfrac{1-2x-x^2}{(1+x^2)^2} \;\Rightarrow$

$$y''=\frac{(1+x^2)^2(-2-2x)-(1-2x-x^2)\cdot 2(1+x^2)(2x)}{[(1+x^2)^2]^2}=\frac{2(1+x^2)[(1+x^2)(-1-x)-(1-2x-x^2)(2x)]}{(1+x^2)^4}$$

$$=\frac{2(-1-x-x^2-x^3-2x+4x^2+2x^3)}{(1+x^2)^3}=\frac{2(x^3+3x^2-3x-1)}{(1+x^2)^3}=\frac{2(x-1)(x^2+4x+1)}{(1+x^2)^3}$$

So $y''=0 \;\Rightarrow\; x=1,-2\pm\sqrt{3}$. Let $a=-2-\sqrt{3}$, $b=-2+\sqrt{3}$, and $c=1$. We can show that $f(a)=\frac{1}{4}\big(1-\sqrt{3}\,\big)$, $f(b)=\frac{1}{4}\big(1+\sqrt{3}\,\big)$, and $f(c)=1$. To show that these three points of inflection lie on one straight line, we'll show that the slopes m_{ac} and m_{bc} are equal.

$$m_{ac}=\frac{f(c)-f(a)}{c-a}=\frac{1-\frac{1}{4}\big(1-\sqrt{3}\,\big)}{1-\big(-2-\sqrt{3}\,\big)}=\frac{\frac{3}{4}+\frac{1}{4}\sqrt{3}}{3+\sqrt{3}}=\frac{1}{4}$$

$$m_{bc}=\frac{f(c)-f(b)}{c-b}=\frac{1-\frac{1}{4}\big(1+\sqrt{3}\,\big)}{1-\big(-2+\sqrt{3}\,\big)}=\frac{\frac{3}{4}-\frac{1}{4}\sqrt{3}}{3-\sqrt{3}}=\frac{1}{4}$$

86. $y=f(x)=e^{-x}\sin x \;\Rightarrow\; y'=e^{-x}\cos x+\sin x(-e^{-x})=e^{-x}(\cos x-\sin x) \;\Rightarrow$

$y''=e^{-x}(-\sin x-\cos x)+(\cos x-\sin x)(-e^{-x})=e^{-x}(-\sin x-\cos x-\cos x+\sin x)=e^{-x}(-2\cos x)$.

So $y''=0 \;\Rightarrow\; \cos x=0 \;\Rightarrow\; x=\frac{\pi}{2}+n\pi$. At these values of x, f has points of inflection and since $\sin\big(\frac{\pi}{2}+n\pi\big)=\pm 1$, we get $y=\pm e^{-x}$, so f intersects the other curves at its inflection points.

Let $g(x)=e^{-x}$ and $h(x)=-e^{-x}$, so that $g'(x)=-e^{-x}$ and $h'(x)=e^{-x}$. Now

$f'\big(\frac{\pi}{2}+n\pi\big)=e^{-(\pi/2+n\pi)}\big[\cos\big(\frac{\pi}{2}+n\pi\big)-\sin\big(\frac{\pi}{2}+n\pi\big)\big]=-e^{-(\pi/2+n\pi)}\sin\big(\frac{\pi}{2}+n\pi\big)$. If n is odd, then

$f'\big(\frac{\pi}{2}+n\pi\big)=e^{-(\pi/2+n\pi)}=h'\big(\frac{\pi}{2}+n\pi\big)$. If n is even, then $f'\big(\frac{\pi}{2}+n\pi\big)=-e^{-(\pi/2+n\pi)}=g'\big(\frac{\pi}{2}+n\pi\big)$.

Thus, at $x=\frac{\pi}{2}+n\pi$, f has the same slope as either g or h, and hence, g and h touch f at its inflection points.

87. $y=x\sin x \;\Rightarrow\; y'=x\cos x+\sin x \;\Rightarrow\; y''=-x\sin x+2\cos x$. $y''=0 \;\Rightarrow\; 2\cos x=x\sin x$ [which is y] $\Rightarrow$

$(2\cos x)^2=(x\sin x)^2 \;\Rightarrow\; 4\cos^2 x=x^2\sin^2 x \;\Rightarrow\; 4\cos^2 x=x^2(1-\cos^2 x) \;\Rightarrow\; 4\cos^2 x+x^2\cos^2 x=x^2 \;\Rightarrow$

$\cos^2 x(4+x^2)=x^2 \;\Rightarrow\; 4\cos^2 x(x^2+4)=4x^2 \;\Rightarrow\; y^2(x^2+4)=4x^2$ since $y=2\cos x$ when $y''=0$.

88. (a) We will make use of the converse of the Concavity Test (along with the stated assumptions); that is, if f is concave upward on I, then $f''>0$ on I. If f and g are CU on I, then $f''>0$ and $g''>0$ on I, so $(f+g)''=f''+g''>0$ on $I \;\Rightarrow$ $f+g$ is CU on I.

(b) Since f is positive and CU on I, $f>0$ and $f''>0$ on I. So $g(x)=[f(x)]^2 \;\Rightarrow\; g'=2ff' \;\Rightarrow$ $g''=2f'f'+2ff''=2(f')^2+2ff''>0 \;\Rightarrow$ g is CU on I.

89. (a) Since f and g are positive, increasing, and CU on I with f'' and g'' never equal to 0, we have $f>0$, $f'\ge 0$, $f''>0$, $g>0$, $g'\ge 0$, $g''>0$ on I. Then $(fg)'=f'g+fg' \;\Rightarrow\; (fg)''=f''g+2f'g'+fg''\ge f''g+fg''>0$ on $I \;\Rightarrow$ fg is CU on I.

(b) In part (a), if f and g are both decreasing instead of increasing, then $f'\le 0$ and $g'\le 0$ on I, so we still have $2f'g'\ge 0$ on I. Thus, $(fg)''=f''g+2f'g'+fg''\ge f''g+fg''>0$ on $I \;\Rightarrow$ fg is CU on I as in part (a).

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(c) Suppose f is increasing and g is decreasing [with f and g positive and CU]. Then $f' \geq 0$ and $g' \leq 0$ on I, so $2f'g' \leq 0$ on I and the argument in parts (a) and (b) fails.

Example 1. $I = (0, \infty)$, $f(x) = x^3$, $g(x) = 1/x$. Then $(fg)(x) = x^2$, so $(fg)'(x) = 2x$ and $(fg)''(x) = 2 > 0$ on I. Thus, fg is CU on I.

Example 2. $I = (0, \infty)$, $f(x) = 4x\sqrt{x}$, $g(x) = 1/x$. Then $(fg)(x) = 4\sqrt{x}$, so $(fg)'(x) = 2/\sqrt{x}$ and $(fg)''(x) = -1/\sqrt{x^3} < 0$ on I. Thus, fg is CD on I.

Example 3. $I = (0, \infty)$, $f(x) = x^2$, $g(x) = 1/x$. Thus, $(fg)(x) = x$, so fg is linear on I.

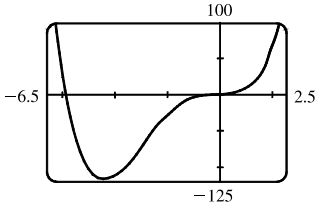
90. Since f and g are CU on $(-\infty, \infty)$, $f'' > 0$ and $g'' > 0$ on $(-\infty, \infty)$. $h(x) = f(g(x)) \;\Rightarrow\; h'(x) = f'(g(x))\, g'(x) \;\Rightarrow\; h''(x) = f''(g(x))\, g'(x)\, g'(x) + f'(g(x))\, g''(x) = f''(g(x))[g'(x)]^2 + f'(g(x))\, g''(x) > 0$ if $f' > 0$. So h is CU if f is increasing.

91. Let the cubic function be $f(x) = ax^3 + bx^2 + cx + d \;\Rightarrow\; f'(x) = 3ax^2 + 2bx + c \;\Rightarrow\; f''(x) = 6ax + 2b$. So f is CU when $6ax + 2b > 0 \;\Leftrightarrow\; x > -b/(3a)$, CD when $x < -b/(3a)$, and so the only point of inflection occurs when $x = -b/(3a)$. If the graph has three x-intercepts x_1, x_2 and x_3, then the expression for $f(x)$ must factor as $f(x) = a(x - x_1)(x - x_2)(x - x_3)$. Multiplying these factors together gives us

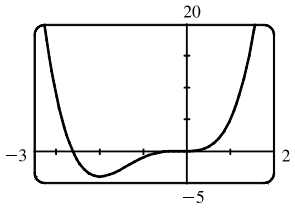
$$f(x) = a[x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3]$$

Equating the coefficients of the x^2-terms for the two forms of f gives us $b = -a(x_1 + x_2 + x_3)$. Hence, the x-coordinate of the point of inflection is $-\dfrac{b}{3a} = -\dfrac{-a(x_1 + x_2 + x_3)}{3a} = \dfrac{x_1 + x_2 + x_3}{3}$.

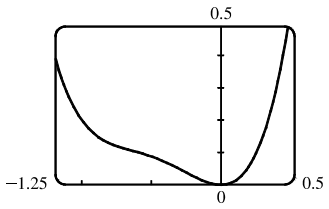
92. $P(x) = x^4 + cx^3 + x^2 \;\Rightarrow\; P'(x) = 4x^3 + 3cx^2 + 2x \;\Rightarrow\; P''(x) = 12x^2 + 6cx + 2$. The graph of $P''(x)$ is a parabola. If $P''(x)$ has two solutions, then it changes sign twice and so has two inflection points. This happens when the discriminant of $P''(x)$ is positive, that is, $(6c)^2 - 4 \cdot 12 \cdot 2 > 0 \;\Leftrightarrow\; 36c^2 - 96 > 0 \;\Leftrightarrow\; |c| > \frac{2\sqrt{6}}{3} \approx 1.63$. If $36c^2 - 96 = 0 \;\Leftrightarrow\; c = \pm\frac{2\sqrt{6}}{3}$, $P''(x)$ is 0 at one point, but there is still no inflection point since $P''(x)$ never changes sign, and if $36c^2 - 96 < 0 \;\Leftrightarrow\; |c| < \frac{2\sqrt{6}}{3}$, then $P''(x)$ never changes sign, and so there is no inflection point.



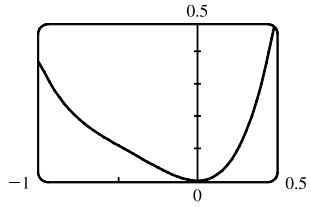
$c = 6$



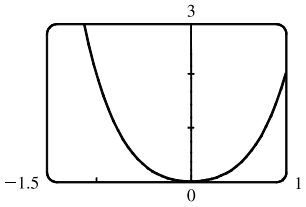
$c = 3$



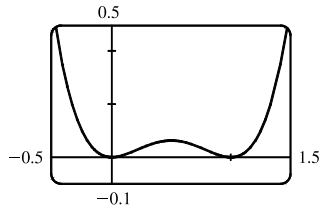
$c = 1.8$



$c = \frac{2\sqrt{6}}{3}$



$c = 0$



$c = -2$

For large positive c, the graph of f has two inflection points and a large dip to the left of the y-axis. As c decreases, the graph

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of f becomes flatter for $x < 0$, and eventually the dip rises above the x-axis, and then disappears entirely, along with the inflection points. As c continues to decrease, the dip and the inflection points reappear, to the right of the origin.

93. By hypothesis $g = f'$ is differentiable on an open interval containing c. Since $(c, f(c))$ is a point of inflection, the concavity changes at $x = c$, so $f''(x)$ changes signs at $x = c$. Hence, by the First Derivative Test, f' has a local extremum at $x = c$. Thus, by Fermat's Theorem $f''(c) = 0$.

94. $f(x) = x^4 \;\Rightarrow\; f'(x) = 4x^3 \;\Rightarrow\; f''(x) = 12x^2 \;\Rightarrow\; f''(0) = 0$. For $x < 0$, $f''(x) > 0$, so f is CU on $(-\infty, 0)$; for $x > 0$, $f''(x) > 0$, so f is also CU on $(0, \infty)$. Since f does not change concavity at 0, $(0, 0)$ is not an inflection point.

95. Using the fact that $|x| = \sqrt{x^2}$, we have that $g(x) = x\,|x| = x\sqrt{x^2} \;\Rightarrow\; g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2\,|x| \;\Rightarrow$ $g''(x) = 2x\big(x^2\big)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$, so $(0, 0)$ is an inflection point. But $g''(0)$ does not exist.

96. There must exist some interval containing c on which f''' is positive, since $f'''(c)$ is positive and f''' is continuous. On this interval, f'' is increasing (since f''' is positive), so $f'' = (f')'$ changes from negative to positive at c. So by the First Derivative Test, f' has a local minimum at $x = c$ and thus cannot change sign there, so f has no maximum or minimum at c. But since f'' changes from negative to positive at c, f has a point of inflection at c (it changes from concave down to concave up).

97. Suppose that f is differentiable on an interval I and $f'(x) > 0$ for all x in I except $x = c$. To show that f is increasing on I, let x_1, x_2 be two numbers in I with $x_1 < x_2$.

Case 1 $x_1 < x_2 < c$. Let J be the interval $\{x \in I \mid x < c\}$. By applying the Increasing/Decreasing Test to f on J, we see that f is increasing on J, so $f(x_1) < f(x_2)$.

Case 2 $c < x_1 < x_2$. Apply the Increasing/Decreasing Test to f on $K = \{x \in I \mid x > c\}$.

Case 3 $x_1 < x_2 = c$. Apply the proof of the Increasing/Decreasing Test, using the Mean Value Theorem (MVT) on the interval $[x_1, x_2]$ and noting that the MVT does not require f to be differentiable at the endpoints of $[x_1, x_2]$.

Case 4 $c = x_1 < x_2$. Same proof as in Case 3.

Case 5 $x_1 < c < x_2$. By Cases 3 and 4, f is increasing on $[x_1, c]$ and on $[c, x_2]$, so $f(x_1) < f(c) < f(x_2)$.

In all cases, we have shown that $f(x_1) < f(x_2)$. Since x_1, x_2 were any numbers in I with $x_1 < x_2$, we have shown that f is increasing on I.

98. $f(x) = cx + \dfrac{1}{x^2+3} \;\Rightarrow\; f'(x) = c - \dfrac{2x}{(x^2+3)^2}$. $f'(x) > 0 \;\Leftrightarrow\; c > \dfrac{2x}{(x^2+3)^2}$ [call this $g(x)$]. Now f' is positive (and hence f increasing) if $c > g$, so we'll find the maximum value of g.

$$g'(x) = \frac{(x^2+3)^2 \cdot 2 - 2x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2} = \frac{2(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \frac{2(3-3x^2)}{(x^2+3)^3} = \frac{6(1+x)(1-x)}{(x^2+3)^3}.$$

$g'(x) = 0 \;\Leftrightarrow\; x = \pm 1$. $g'(x) > 0$ on $(0, 1)$ and $g'(x) < 0$ on $(1, \infty)$, so g is increasing on $(0, 1)$ and decreasing on

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$(1, \infty)$, and hence g has a maximum value on $(0, \infty)$ of $g(1) = \frac{2}{16} = \frac{1}{8}$. Also since $g(x) \le 0$ if $x \le 0$, the maximum value of g on $(-\infty, \infty)$ is $\frac{1}{8}$. Thus, when $c > \frac{1}{8}$, f is increasing. When $c = \frac{1}{8}$, $f'(x) > 0$ on $(-\infty, 1)$ and $(1, \infty)$, and hence f is increasing on these intervals. Since f is continuous, we may conclude that f is also increasing on $(-\infty, \infty)$ if $c = \frac{1}{8}$. Therefore, f is increasing on $(-\infty, \infty)$ if $c \ge \frac{1}{8}$.

99. (a) $f(x) = x^4 \sin\dfrac{1}{x} \quad \Rightarrow \quad f'(x) = x^4 \cos\dfrac{1}{x}\left(-\dfrac{1}{x^2}\right) + \sin\dfrac{1}{x}\,(4x^3) = 4x^3 \sin\dfrac{1}{x} - x^2 \cos\dfrac{1}{x}$.

$g(x) = x^4\left(2 + \sin\dfrac{1}{x}\right) = 2x^4 + f(x) \quad \Rightarrow \quad g'(x) = 8x^3 + f'(x)$.

$h(x) = x^4\left(-2 + \sin\dfrac{1}{x}\right) = -2x^4 + f(x) \quad \Rightarrow \quad h'(x) = -8x^3 + f'(x)$.

It is given that $f(0) = 0$, so $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^4 \sin\dfrac{1}{x} - 0}{x} = \lim\limits_{x \to 0} x^3 \sin\dfrac{1}{x}$. Since $-\left|x^3\right| \le x^3 \sin\dfrac{1}{x} \le \left|x^3\right|$ and $\lim\limits_{x \to 0} \left|x^3\right| = 0$, we see that $f'(0) = 0$ by the Squeeze Theorem. Also, $g'(0) = 8(0)^3 + f'(0) = 0$ and $h'(0) = -8(0)^3 + f'(0) = 0$, so 0 is a critical number of f, g, and h.

For $x_{2n} = \dfrac{1}{2n\pi}$ [n a nonzero integer], $\sin\dfrac{1}{x_{2n}} = \sin 2n\pi = 0$ and $\cos\dfrac{1}{x_{2n}} = \cos 2n\pi = 1$, so $f'(x_{2n}) = -x_{2n}^2 < 0$.

For $x_{2n+1} = \dfrac{1}{(2n+1)\pi}$, $\sin\dfrac{1}{x_{2n+1}} = \sin(2n+1)\pi = 0$ and $\cos\dfrac{1}{x_{2n+1}} = \cos(2n+1)\pi = -1$, so $f'(x_{2n+1}) = x_{2n+1}^2 > 0$. Thus, f' changes sign infinitely often on both sides of 0.

Next, $g'(x_{2n}) = 8x_{2n}^3 + f'(x_{2n}) = 8x_{2n}^3 - x_{2n}^2 = x_{2n}^2(8x_{2n} - 1) < 0$ for $x_{2n} < \frac{1}{8}$, but $g'(x_{2n+1}) = 8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(8x_{2n+1} + 1) > 0$ for $x_{2n+1} > -\frac{1}{8}$, so g' changes sign infinitely often on both sides of 0.

Last, $h'(x_{2n}) = -8x_{2n}^3 + f'(x_{2n}) = -8x_{2n}^3 - x_{2n}^2 = -x_{2n}^2(8x_{2n} + 1) < 0$ for $x_{2n} > -\frac{1}{8}$ and $h'(x_{2n+1}) = -8x_{2n+1}^3 + x_{2n+1}^2 = x_{2n+1}^2(-8x_{2n+1} + 1) > 0$ for $x_{2n+1} < \frac{1}{8}$, so h' changes sign infinitely often on both sides of 0.

(b) $f(0) = 0$ and since $\sin\dfrac{1}{x}$ and hence $x^4 \sin\dfrac{1}{x}$ is both positive and negative inifinitely often on both sides of 0, and arbitrarily close to 0, f has neither a local maximum nor a local minimum at 0.

Since $2 + \sin\dfrac{1}{x} \ge 1$, $g(x) = x^4\left(2 + \sin\dfrac{1}{x}\right) > 0$ for $x \ne 0$, so $g(0) = 0$ is a local minimum.

Since $-2 + \sin\dfrac{1}{x} \le -1$, $h(x) = x^4\left(-2 + \sin\dfrac{1}{x}\right) < 0$ for $x \ne 0$, so $h(0) = 0$ is a local maximum.

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4.4 Indeterminate Forms and l'Hospital's Rule

Note: The use of l'Hospital's Rule is indicated by an H above the equal sign: $\overset{\text{H}}{=}$

1. (a) $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is an indeterminate form of type $\dfrac{0}{0}$.

(b) $\lim\limits_{x\to a}\dfrac{f(x)}{p(x)}=0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim\limits_{x\to a}\dfrac{h(x)}{p(x)}=0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim\limits_{x\to a}p(x)=\infty$ and $f(x)\to 0$ through positive values, then $\lim\limits_{x\to a}\dfrac{p(x)}{f(x)}=\infty$. [For example, take $a=0$, $p(x)=1/x^2$, and $f(x)=x^2$.] If $f(x)\to 0$ through negative values, then $\lim\limits_{x\to a}\dfrac{p(x)}{f(x)}=-\infty$. [For example, take $a=0$, $p(x)=1/x^2$, and $f(x)=-x^2$.] If $f(x)\to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a=0$, $p(x)=1/x^2$, and $f(x)=x$.]

(e) $\lim\limits_{x\to a}\dfrac{p(x)}{q(x)}$ is an indeterminate form of type $\dfrac{\infty}{\infty}$.

2. (a) $\lim\limits_{x\to a}[f(x)p(x)]$ is an indeterminate form of type $0\cdot\infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim\limits_{x\to a}[h(x)p(x)]=\infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim\limits_{x\to a}[p(x)q(x)]=\infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x)-p(x)$ is large negative. Thus, $\lim\limits_{x\to a}[f(x)-p(x)]=-\infty$.

(b) $\lim\limits_{x\to a}[\,p(x)-q(x)]$ is an indeterminate form of type $\infty-\infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)+q(x)$ is large. Thus, $\lim\limits_{x\to a}[\,p(x)+q(x)]=\infty$.

4. (a) $\lim\limits_{x\to a}[f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y=[f(x)]^{p(x)}$, then $\ln y=p(x)\ln f(x)$. When x is near a, $p(x)\to\infty$ and $\ln f(x)\to-\infty$, so $\ln y\to-\infty$. Therefore, $\lim\limits_{x\to a}[f(x)]^{p(x)}=\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=0$, provided f^p is defined.

(c) $\lim\limits_{x\to a}[h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim\limits_{x\to a}[\,p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y=[\,p(x)]^{q(x)}$, then $\ln y=q(x)\ln p(x)$. When x is near a, $q(x)\to\infty$ and $\ln p(x)\to\infty$, so $\ln y\to\infty$. Therefore, $\lim\limits_{x\to a}[\,p(x)]^{q(x)}=\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=\infty$.

(f) $\lim\limits_{x\to a}\sqrt[q(x)]{p(x)}=\lim\limits_{x\to a}[\,p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

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5. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.8}{\frac{4}{5}} = \frac{9}{4}$$

6. From the graphs of f and g, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.5}{-1} = -\frac{3}{2}$$

7. f and $g = e^x - 1$ are differentiable and $g' = e^x \neq 0$ on an open interval that contains 0. $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} g(x) = 0$, so we have the indeterminate form $\frac{0}{0}$ and can apply l'Hospital's Rule.

$$\lim_{x\to 0}\frac{f(x)}{e^x - 1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{f'(x)}{e^x} = \frac{1}{1} = 1$$

Note that $\lim_{x\to 0} f'(x) = 1$ since the graph of f has the same slope as the line $y = x$ at $x = 0$.

8. This limit has the form $\frac{0}{0}$. $\lim_{x\to 3}\dfrac{x-3}{x^2-9} = \lim_{x\to 3}\dfrac{x-3}{(x+3)(x-3)} = \lim_{x\to 3}\dfrac{1}{x+3} = \dfrac{1}{3+3} = \dfrac{1}{6}$

Note: Alternatively, we could apply l'Hospital's Rule.

9. This limit has the form $\frac{0}{0}$. $\lim_{x\to 4}\dfrac{x^2-2x-8}{x-4} = \lim_{x\to 4}\dfrac{(x-4)(x+2)}{x-4} = \lim_{x\to 4}(x+2) = 4+2 = 6$

Note: Alternatively, we could apply l'Hospital's Rule.

10. This limit has the form $\frac{0}{0}$. $\lim_{x\to -2}\dfrac{x^3+8}{x+2} \overset{\text{H}}{=} \lim_{x\to -2}\dfrac{3x^2}{1} = 3(-2)^2 = 12$

Note: Alternatively, we could factor and simplify.

11. This limit has the form $\frac{0}{0}$. $\lim_{x\to 1}\dfrac{x^7-1}{x^3-1} \overset{\text{H}}{=} \lim_{x\to 1}\dfrac{7x^6}{3x^2} = \dfrac{7}{3}$

Note: Alternatively, we could factor and simplify.

12. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\cdot\frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4}\frac{x-4}{(x-4)\left(\sqrt{x}+2\right)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

Note: Alternatively, we could apply l'Hospital's Rule.

13. This limit has the form $\frac{0}{0}$. $\lim_{x\to \pi/4}\dfrac{\sin x - \cos x}{\tan x - 1} \overset{\text{H}}{=} \lim_{x\to \pi/4}\dfrac{\cos x + \sin x}{\sec^2 x} = \lim_{x\to 4}\dfrac{\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}}{\left(\sqrt{2}\right)^2} = \dfrac{\sqrt{2}}{2}$

14. This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\dfrac{\tan 3x}{\sin 2x} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{3\sec^2 3x}{2\cos 2x} = \dfrac{3(1)^2}{2(1)} = \dfrac{3}{2}$

15. This limit has the form $\frac{0}{0}$. $\lim_{t\to 0}\dfrac{e^{2t}-1}{\sin t} \overset{\text{H}}{=} \lim_{t\to 0}\dfrac{2e^{2t}}{\cos t} = \dfrac{2(1)}{1} = 2$

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16. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to0}\dfrac{x^2}{1-\cos x}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{2x}{\sin x}=\lim\limits_{x\to0}\dfrac{2}{(\sin x)/x}=\dfrac{2}{1}=2$

17. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to1}\dfrac{\sin(x-1)}{x^3+x-2}\overset{\text{H}}{=}\lim\limits_{x\to1}\dfrac{\cos(x-1)}{3x^2+1}=\dfrac{\cos0}{3(1)^2+1}=\dfrac{1}{4}$

18. The limit can be evaluated by substituting π for θ. $\lim\limits_{\theta\to\pi}\dfrac{1+\cos\theta}{1-\cos\theta}=\dfrac{1+(-1)}{1-(-1)}=\dfrac{0}{2}=0$

19. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x}}{1+e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{\frac{1}{2\sqrt{x}}}{e^x}=\lim\limits_{x\to\infty}\dfrac{1}{e^x\cdot2\sqrt{x}}=0$

20. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1+2x}{-4x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2}{-4}=-\dfrac{1}{2}$.

A better method is to divide the numerator and the denominator by x^2: $\lim\limits_{x\to\infty}\dfrac{x+x^2}{1-2x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}+1}{\frac{1}{x^2}-2}=\dfrac{0+1}{0-2}=-\dfrac{1}{2}$.

21. $\lim\limits_{x\to0^+}[(\ln x)/x]=-\infty$ since $\ln x\to-\infty$ as $x\to0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

22. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln\sqrt{x}}{x^2}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{2}\ln x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}=\dfrac{\frac{1}{2x}}{2x}=\lim\limits_{x\to\infty}\dfrac{1}{4x^2}=0$

23. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to3}\dfrac{\ln(x/3)}{3-x}\overset{\text{H}}{=}\lim\limits_{x\to3}\dfrac{1/(x/3)\cdot(1/3)}{-1}=\lim\limits_{x\to3}\left(-\dfrac{1}{x}\right)=-\dfrac{1}{3}$

24. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to0}\dfrac{8^t-5^t}{t}\overset{\text{H}}{=}\lim\limits_{t\to0}\dfrac{8^t\ln8-5^t\ln5}{1}=\ln8-\ln5=\ln\dfrac{8}{5}$

25. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to0}\frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}\overset{\text{H}}{=}\lim_{x\to0}\frac{\frac{1}{2}(1+2x)^{-1/2}\cdot2-\frac{1}{2}(1-4x)^{-1/2}(-4)}{1}$$
$$=\lim_{x\to0}\left(\frac{1}{\sqrt{1+2x}}+\frac{2}{\sqrt{1-4x}}\right)=\frac{1}{\sqrt{1}}+\frac{2}{\sqrt{1}}=3$$

26. This limit has the form $\frac{\infty}{\infty}$.

$$\lim_{u\to\infty}\frac{e^{u/10}}{u^3}\overset{\text{H}}{=}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{3u^2}\overset{\text{H}}{=}\frac{1}{30}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{2u}\overset{\text{H}}{=}\frac{1}{600}\lim_{u\to\infty}\frac{e^{u/10}\cdot\frac{1}{10}}{1}=\frac{1}{6000}\lim_{u\to\infty}e^{u/10}=\infty$$

27. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to0}\dfrac{e^x+e^{-x}-2}{e^x-x-1}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{e^x-e^{-x}}{e^x-1}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{e^x+e^{-x}}{e^x}=\dfrac{1+1}{1}=2$

28. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to0}\dfrac{\sinh x-x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{\cosh x-1}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{\sinh x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{\cosh x}{6}=\dfrac{1}{6}$

29. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to0}\dfrac{\tanh x}{\tan x}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{\operatorname{sech}^2x}{\sec^2x}=\dfrac{\operatorname{sech}^2 0}{\sec^2 0}=\dfrac{1}{1}=1$

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30. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x-\sin x}{x-\tan x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{1-\cos x}{1-\sec^2 x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-(-\sin x)}{-2\sec x\,(\sec x\,\tan x)} = -\frac{1}{2}\lim_{x\to 0}\frac{\sin x\left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x}$$
$$= -\tfrac{1}{2}\lim_{x\to 0}\cos^3 x = -\tfrac{1}{2}(1)^3 = -\tfrac{1}{2}$$

Another method is to write the limit as $\lim_{x\to 0}\dfrac{1-\dfrac{\sin x}{x}}{1-\dfrac{\tan x}{x}}$.

31. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x}{x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{1/\sqrt{1-x^2}}{1} = \lim_{x\to 0}\frac{1}{\sqrt{1-x^2}} = \frac{1}{1} = 1$

32. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{(\ln x)^2}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2(\ln x)(1/x)}{1} = 2\lim_{x\to\infty}\frac{\ln x}{x} \overset{\text{H}}{=} 2\lim_{x\to\infty}\frac{1/x}{1} = 2(0) = 0$

33. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{x3^x}{3^x-1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{x3^x\ln 3+3^x}{3^x\ln 3} = \lim_{x\to 0}\frac{3^x(x\ln 3+1)}{3^x\ln 3} = \lim_{x\to 0}\frac{x\ln 3+1}{\ln 3} = \frac{1}{\ln 3}$

34. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{e^x+e^{-x}-2\cos x}{x\sin x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x-e^{-x}+2\sin x}{x\cos x+\sin x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{e^x+e^{-x}+2\cos x}{-x\sin x+\cos x+\cos x} = \frac{1+1+2}{0+1+1} = \frac{4}{2} = 2$$

35. This limit can be evaluated by substituting 0 for x. $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{\cos x+e^x-1} = \frac{\ln 1}{1+1-1} = \frac{0}{1} = 0$

36. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x\sin(x-1)}{2x^2-x-1} \overset{\text{H}}{=} \lim_{x\to 1}\frac{x\cos(x-1)+\sin(x-1)}{4x-1} = \frac{\cos 0}{4-1} = \frac{1}{3}$

37. This limit has the form $\frac{0}{\infty}$, so l'Hospital's Rule doesn't apply. As $x\to 0^+$, $\arctan 2x\to 0$ and $\ln x\to -\infty$, so $\displaystyle\lim_{x\to 0^+}\frac{\arctan 2x}{\ln x} = 0$.

38. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{x^2\sin x}{\sin x-x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{x^2\cos x+2x\sin x}{\cos x-1} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-x^2\sin x+2x\cos x+2x\cos x+2\sin x}{-\sin x}$$
$$= \lim_{x\to 0}\frac{(2-x^2)\sin x+4x\cos x}{-\sin x}$$
$$\overset{\text{H}}{=} \lim_{x\to 0}\frac{(2-x^2)\cos x-2x\sin x-4x\sin x+4\cos x}{-\cos x} = \frac{2(1)-0-0+4(1)}{-1} = -6$$

39. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 1}\frac{x^a-1}{x^b-1}$ [for $b\neq 0$] $\displaystyle\overset{\text{H}}{=}\lim_{x\to 1}\frac{ax^{a-1}}{bx^{b-1}} = \frac{a(1)}{b(1)} = \frac{a}{b}$

40. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to\infty}\frac{e^{-x}}{(\pi/2)-\tan^{-1}x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{-e^{-x}}{-\dfrac{1}{1+x^2}} = \lim_{x\to\infty}\frac{1+x^2}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2x}{e^x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{2}{e^x} = 0$

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41. This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\cos x-1+\frac{1}{2}x^2}{x^4}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\sin x+x}{4x^3}\overset{\text{H}}{=}\lim_{x\to 0}\frac{-\cos x+1}{12x^2}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{24x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{24}=\frac{1}{24}$$

42. This limit has the form $\frac{0}{0}$.

$$\begin{aligned}\lim_{x\to 0}\frac{x-\sin x}{x\sin(x^2)}&\overset{\text{H}}{=}\lim_{x\to 0}\frac{1-\cos x}{x\cos(x^2)\cdot 2x+\sin(x^2)}=\lim_{x\to 0}\frac{1-\cos x}{2x^2\cos(x^2)+\sin(x^2)}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{2x^2\left[-\sin(x^2)\cdot 2x\right]+\cos(x^2)\cdot 4x+\cos(x^2)\cdot 2x}=\lim_{x\to 0}\frac{\sin x}{6x\cos(x^2)-4x^3\sin(x^2)}\\&\overset{\text{H}}{=}\lim_{x\to 0}\frac{\cos x}{6x\left[-\sin(x^2)\cdot 2x\right]+\cos(x^2)\cdot 6-4x^3\cos(x^2)\cdot 2x-\sin(x^2)\cdot 12x^2}=\frac{1}{0+1\cdot 6-0-0}=\frac{1}{6}\end{aligned}$$

43. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to\infty}x\sin(\pi/x)=\lim_{x\to\infty}\frac{\sin(\pi/x)}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2}=\pi\lim_{x\to\infty}\cos(\pi/x)=\pi(1)=\pi$$

44. This limit has the form $\infty\cdot 0$. We'll change it to the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\sqrt{x}\,e^{-x/2}=\lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}}=\lim_{x\to\infty}\frac{1}{\sqrt{x}\,e^{x/2}}=0$$

45. This limit has the form $0\cdot\infty$. We'll change it to the form $\frac{0}{0}$. $\lim_{x\to 0}\sin 5x\csc 3x=\lim_{x\to 0}\frac{\sin 5x}{\sin 3x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{5\cos 5x}{3\cos 3x}=\frac{5\cdot 1}{3\cdot 1}=\frac{5}{3}$

46. This limit has the form $(-\infty)\cdot 0$.

$$\lim_{x\to-\infty}x\ln\left(1-\frac{1}{x}\right)=\lim_{x\to-\infty}\frac{\ln\left(1-\frac{1}{x}\right)}{\frac{1}{x}}\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{\frac{1}{1-1/x}\cdot\frac{1}{x^2}}{-\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{-1}{1-\frac{1}{x}}=\frac{-1}{1}=-1$$

47. This limit has the form $\infty\cdot 0$. $\lim_{x\to\infty}x^3e^{-x^2}=\lim_{x\to\infty}\frac{x^3}{e^{x^2}}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3x^2}{2xe^{x^2}}=\lim_{x\to\infty}\frac{3x}{2e^{x^2}}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{3}{4xe^{x^2}}=0$

48. This limit has the form $\infty\cdot 0$.

$$\lim_{x\to\infty}x^{3/2}\sin(1/x)=\lim_{x\to\infty}x^{1/2}\cdot\frac{\sin(1/x)}{1/x}=\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\frac{\sin t}{t}\quad[\text{where } t=1/x]\quad=\infty \text{ since as } t\to 0^+,\ \frac{1}{\sqrt{t}}\to\infty$$

and $\dfrac{\sin t}{t}\to 1$.

49. This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to 1^+}\ln x\,\tan(\pi x/2)=\lim_{x\to 1^+}\frac{\ln x}{\cot(\pi x/2)}\overset{\text{H}}{=}\lim_{x\to 1^+}\frac{1/x}{(-\pi/2)\csc^2(\pi x/2)}=\frac{1}{(-\pi/2)(1)^2}=-\frac{2}{\pi}$$

50. This limit has the form $0\cdot\infty$. $\lim_{x\to(\pi/2)^-}\cos x\,\sec 5x=\lim_{x\to(\pi/2)^-}\frac{\cos x}{\cos 5x}\overset{\text{H}}{=}\lim_{x\to(\pi/2)^-}\frac{-\sin x}{-5\sin 5x}=\frac{-1}{-5}=\frac{1}{5}$

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51. This limit has the form $\infty - \infty$.

$$\lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)=\lim_{x\to 1}\frac{x\ln x-(x-1)}{(x-1)\ln x}\overset{\text{H}}{=}\lim_{x\to 1}\frac{x(1/x)+\ln x-1}{(x-1)(1/x)+\ln x}=\lim_{x\to 1}\frac{\ln x}{1-(1/x)+\ln x}$$

$$\overset{\text{H}}{=}\lim_{x\to 1}\frac{1/x}{1/x^2+1/x}\cdot\frac{x^2}{x^2}=\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{1+1}=\frac{1}{2}$$

52. This limit has the form $\infty - \infty$. $\lim_{x\to 0}(\csc x-\cot x)=\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right)=\lim_{x\to 0}\frac{1-\cos x}{\sin x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{\sin x}{\cos x}=0$

53. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0^+}\frac{e^x-1-x}{x(e^x-1)}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{e^x-1}{xe^x+e^x-1}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{e^x}{xe^x+e^x+e^x}=\frac{1}{0+1+1}=\frac{1}{2}$$

54. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan^{-1}x}\right)=\lim_{x\to 0^+}\frac{\tan^{-1}x-x}{x\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/(1+x^2)-1}{x/(1+x^2)+\tan^{-1}x}=\lim_{x\to 0^+}\frac{1-(1+x^2)}{x+(1+x^2)\tan^{-1}x}$$

$$=\lim_{x\to 0^+}\frac{-x^2}{x+(1+x^2)\tan^{-1}x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{-2x}{1+(1+x^2)(1/(1+x^2))+(\tan^{-1}x)(2x)}$$

$$=\lim_{x\to 0^+}\frac{-2x}{2+2x\tan^{-1}x}=\frac{0}{2+0}=0$$

55. This limit has the form $\infty - \infty$.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\tan x}\right)=\lim_{x\to 0^+}\frac{\tan x-x}{x\tan x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{\sec^2x-1}{x\sec^2x+\tan x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{2\sec x\cdot\sec x\tan x}{x\cdot 2\sec x\cdot\sec x\tan x+\sec^2x+\sec^2x}$$

$$=\frac{0}{0+1+1}=0$$

56. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}(x-\ln x)=\lim_{x\to\infty}x\left(1-\frac{\ln x}{x}\right)=\infty \text{ since } \lim_{x\to\infty}\frac{\ln x}{x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{1/x}{1}=0.$$

57. $y=x^{\sqrt{x}}\quad\Rightarrow\quad \ln y=\sqrt{x}\,\ln x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}\sqrt{x}\ln x=\lim_{x\to 0^+}\frac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}=-2\lim_{x\to 0^+}\sqrt{x}=0\quad\Rightarrow$$

$$\lim_{x\to 0^+}x^{\sqrt{x}}=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

58. $y=(\tan 2x)^x\quad\Rightarrow\quad \ln y=x\cdot\ln\tan 2x$, so

$$\lim_{x\to 0^+}\ln y=\lim_{x\to 0^+}x\cdot\ln\tan 2x=\lim_{x\to 0^+}\frac{\ln\tan 2x}{1/x}\overset{\text{H}}{=}\lim_{x\to 0^+}\frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2}=\lim_{x\to 0^+}\frac{-2x^2\cos 2x}{\sin 2x\,\cos^2 2x}$$

$$=\lim_{x\to 0^+}\frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+}\frac{-x}{\cos 2x}=1\cdot 0=0\quad\Rightarrow$$

$$\lim_{x\to 0^+}(\tan 2x)^x=\lim_{x\to 0^+}e^{\ln y}=e^0=1.$$

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59. $y = (1-2x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{-2/(1-2x)}{1} = -2 \quad \Rightarrow$

$\lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\ln y} = e^{-2}$.

60. $y = \left(1+\dfrac{a}{x}\right)^{bx} \quad \Rightarrow \quad \ln y = bx\ln\left(1+\dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1+a/x} = ab \quad \Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

61. $y = x^{1/(1-x)} \quad \Rightarrow \quad \ln y = \dfrac{1}{1-x}\ln x$, so $\lim\limits_{x\to 1^+} \ln y = \lim\limits_{x\to 1^+} \dfrac{1}{1-x}\ln x = \lim\limits_{x\to 1^+} \dfrac{\ln x}{1-x} \overset{\text{H}}{=} \lim\limits_{x\to 1^+} \dfrac{1/x}{-1} = -1 \quad \Rightarrow$

$\lim\limits_{x\to 1^+} x^{1/(1-x)} = \lim\limits_{x\to 1^+} e^{\ln y} = e^{-1} = \dfrac{1}{e}$.

62. $y = (e^x+10x)^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln(e^x+10x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{1}{x}\ln(e^x+10x) = \lim_{x\to\infty} \frac{\ln(e^x+10x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{e^x+10x}\cdot(e^x+10)}{1}$$

$$= \lim_{x\to\infty} \frac{e^x+10}{e^x+10x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x+10} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = \lim_{x\to\infty} (1) = 1 \quad \Rightarrow$$

$$\lim_{x\to\infty} (e^x+10x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e$$

63. $y = x^{1/x} \quad \Rightarrow \quad \ln y = (1/x)\ln x \quad \Rightarrow \quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0 \quad \Rightarrow$

$\lim\limits_{x\to\infty} x^{1/x} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$

64. $y = x^{e^{-x}} \quad \Rightarrow \quad \ln y = e^{-x}\ln x \quad \Rightarrow \quad \lim\limits_{x\to\infty} \ln y = \lim\limits_{x\to\infty} \dfrac{\ln x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{e^x} = \lim\limits_{x\to\infty} \dfrac{1}{xe^x} = 0 \quad \Rightarrow$

$\lim\limits_{x\to\infty} x^{e^{-x}} = \lim\limits_{x\to\infty} e^{\ln y} = e^0 = 1$

65. $y = (4x+1)^{\cot x} \quad \Rightarrow \quad \ln y = \cot x\,\ln(4x+1)$, so $\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} \dfrac{\ln(4x+1)}{\tan x} \overset{\text{H}}{=} \lim\limits_{x\to 0^+} \dfrac{\dfrac{4}{4x+1}}{\sec^2 x} = 4 \quad \Rightarrow$

$\lim\limits_{x\to 0^+} (4x+1)^{\cot x} = \lim\limits_{x\to 0^+} e^{\ln y} = e^4$.

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66. $y=(1-\cos x)^{\sin x} \quad \Rightarrow \quad \ln y=\sin x\ln(1-\cos x)$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\sin x\ln(1-\cos x)=\lim_{x\to0^+}\frac{\ln(1-\cos x)}{\csc x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{\dfrac{1}{1-\cos x}\cdot\sin x}{-\csc x\cot x}$$

$$=-\lim_{x\to0^+}\frac{\sin x}{(1-\cos x)\csc x\cot x}=-\lim_{x\to0^+}\frac{\sin x}{\csc x\cot x-\cot^2x}\cdot\left(\frac{\sin^2x}{\sin^2x}\right)$$

$$=-\lim_{x\to0^+}\frac{\sin^3x}{\cos x-\cos^2x}=-\lim_{x\to0^+}\frac{\sin^3x}{(1-\cos x)\cos x}$$

$$\overset{\text{H}}{=}-\lim_{x\to0^+}\frac{3\sin^2x\cos x}{(1-\cos x)(-\sin x)+\cos x\,(\sin x)}=-\lim_{x\to0^+}\frac{3\sin x\cos x}{(\cos x-1)+\cos x}=-\frac{0}{0+1}=0 \quad \Rightarrow$$

$$\lim_{x\to0^+}(1-\cos x)^{\sin x}=\lim_{x\to0^+}e^{\ln y}=e^0=1$$

67. $y=(1+\sin3x)^{1/x} \quad \Rightarrow \quad \ln y=\dfrac{1}{x}\ln(1+\sin3x) \quad \Rightarrow$

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln(1+\sin3x)}{x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{[1/(1+\sin3x)]\cdot3\cos3x}{1}=\lim_{x\to0^+}\frac{3\cos3x}{1+\sin3x}=\frac{3\cdot1}{1+0}=3 \quad \Rightarrow$$

$$\lim_{x\to0^+}(1+\sin3x)^{1/x}=\lim_{x\to0^+}e^{\ln y}=e^3$$

68. $y=(\cos x)^{1/x^2} \quad \Rightarrow \quad \ln y=\dfrac{1}{x^2}\ln\cos x$, so

$$\lim_{x\to0}\ln y=\lim_{x\to0}\frac{1}{x^2}\ln\cos x=\lim_{x\to0}\frac{\ln\cos x}{x^2}\overset{\text{H}}{=}\lim_{x\to0}\frac{\dfrac{1}{\cos x}\cdot(-\sin x)}{2x}=\lim_{x\to0}\frac{-\tan x}{2x}\overset{\text{H}}{=}\lim_{x\to0}\frac{-\sec^2x}{2}=-\frac{1}{2} \quad \Rightarrow$$

$$\lim_{x\to0}(\cos x)^{1/x^2}=\lim_{x\to0}e^{\ln y}=e^{-1/2}=\frac{1}{\sqrt{e}}$$

69. The given limit is $\displaystyle\lim_{x\to0^+}\frac{x^x-1}{\ln x+x-1}$. Note that $y=x^x \quad \Rightarrow \quad \ln y=x\ln x$, so

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}x\ln x=\lim_{x\to0^+}\frac{\ln x}{\dfrac{1}{x}}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{\dfrac{1}{x}}{-\dfrac{1}{x^2}}=\lim_{x\to0^+}(-x)=0 \quad \Rightarrow \quad \lim_{x\to0^+}x^x=\lim_{x\to0^+}e^{\ln y}=e^0=1.$$

Therefore, the numerator of the given limit has limit $1-1=0$ as $x\to0^+$. The denominator of the given limit $\to-\infty$ as $x\to0^+$ since $\ln x\to-\infty$ as $x\to0^+$. Thus, $\displaystyle\lim_{x\to0^+}\frac{x^x-1}{\ln x+x-1}=0$.

70. $y=\left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \quad \Rightarrow \quad \ln y=(2x+1)\,\ln\left(\dfrac{2x-3}{2x+5}\right) \quad \Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}=\lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$=\lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)}=-8 \quad \Rightarrow \quad \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=\lim_{x\to\infty}e^{\ln y}=e^{-8}$$

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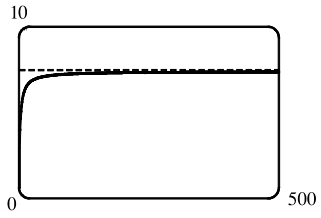
71.



From the graph, if $x = 500$, $y \approx 7.36$. The limit has the form 1^∞.

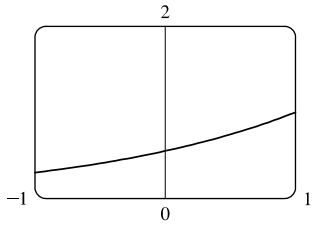
Now $y = \left(1 + \dfrac{2}{x}\right)^x \quad \Rightarrow \quad \ln y = x \ln\left(1 + \dfrac{2}{x}\right) \quad \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(1+2/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\dfrac{1}{1+2/x}\left(-\dfrac{2}{x^2}\right)}{-1/x^2}$$

$$= 2 \lim_{x\to\infty} \frac{1}{1+2/x} = 2(1) = 2 \quad \Rightarrow$$

$$\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^2 \ [\approx 7.39]$$

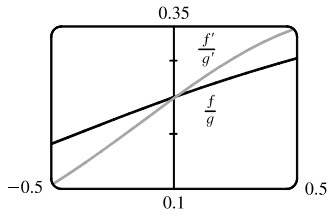
72.



From the graph, as $x \to 0$, $y \approx 0.55$. The limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0} \frac{5^x - 4^x}{3^x - 2^x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{5^x \ln 5 - 4^x \ln 4}{3^x \ln 3 - 2^x \ln 2} = \frac{\ln 5 - \ln 4}{\ln 3 - \ln 2} = \frac{\ln \frac{5}{4}}{\ln \frac{3}{2}} \ [\approx 0.55]$$

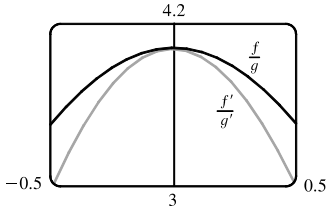
73.



From the graph, it appears that $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{f'(x)}{g'(x)} = 0.25$.

We calculate $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{e^x - 1}{x^3 + 4x} \overset{\text{H}}{=} \lim_{x\to 0} \dfrac{e^x}{3x^2 + 4} = \dfrac{1}{4}$.

74.



From the graph, it appears that $\lim_{x\to 0} \dfrac{f(x)}{g(x)} = \lim_{x\to 0} \dfrac{f'(x)}{g'(x)} = 4$. We calculate

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{2x \sin x}{\sec x - 1} \overset{\text{H}}{=} \lim_{x\to 0} \frac{2(x\cos x + \sin x)}{\sec x \tan x}$$

$$\overset{\text{H}}{=} \lim_{x\to 0} \frac{2(-x\sin x + \cos x + \cos x)}{\sec x(\sec^2 x) + \tan x(\sec x \tan x)} = \frac{4}{1} = 4$$

75. $\lim_{x\to\infty} \dfrac{e^x}{x^n} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{e^x}{nx^{n-1}} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{e^x}{n(n-1)x^{n-2}} \overset{\text{H}}{=} \cdots \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{e^x}{n!} = \infty$

76. This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty} \dfrac{\ln x}{x^p} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{1/x}{px^{p-1}} = \lim_{x\to\infty} \dfrac{1}{px^p} = 0$ since $p > 0$.

77. $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{1}{\frac{1}{2}(x^2+1)^{-1/2}(2x)} = \lim_{x\to\infty} \dfrac{\sqrt{x^2+1}}{x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to try dividing the numerator and denominator by x: $\lim_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \dfrac{x/x}{\sqrt{x^2/x^2 + 1/x^2}} = \lim_{x\to\infty} \dfrac{1}{\sqrt{1+1/x^2}} = \dfrac{1}{1} = 1$

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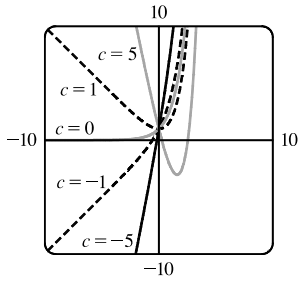
78. $\lim\limits_{x\to(\pi/2)^-}\dfrac{\sec x}{\tan x}\overset{\text{H}}{=}\lim\limits_{x\to(\pi/2)^-}\dfrac{\sec x\,\tan x}{\sec^2 x}=\lim\limits_{x\to(\pi/2)^-}\dfrac{\tan x}{\sec x}$. Repeated applications of l'Hospital's Rule result in the original limit or the limit of the reciprocal of the function. Another method is to simplify first:

$$\lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan x}=\lim_{x\to(\pi/2)^-}\frac{1/\cos x}{\sin x/\cos x}=\lim_{x\to(\pi/2)^-}\frac{1}{\sin x}=\frac{1}{1}=1$$

79. $f(x)=e^x-cx \;\Rightarrow\; f'(x)=e^x-c=0 \;\Leftrightarrow\; e^x=c \;\Leftrightarrow\; x=\ln c,\ c>0$. $f''(x)=e^x>0$, so f is CU on $(-\infty,\infty)$. $\lim\limits_{x\to\infty}(e^x-cx)=\lim\limits_{x\to\infty}\left[x\left(\dfrac{e^x}{x}-c\right)\right]=L_1$. Now $\lim\limits_{x\to\infty}\dfrac{e^x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{1}=\infty$, so $L_1=\infty$, regardless of the value of c. For $L=\lim\limits_{x\to-\infty}(e^x-cx)$, $e^x\to0$, so L is determined by $-cx$. If $c>0$, $-cx\to\infty$, and $L=\infty$. If $c<0$, $-cx\to-\infty$, and $L=-\infty$. Thus, f has an absolute minimum for $c>0$. As c increases, the minimum points $(\ln c, c-c\ln c)$, get farther away from the origin.



80. (a) $\lim\limits_{t\to\infty}v=\lim\limits_{t\to\infty}\dfrac{mg}{c}\left(1-e^{-ct/m}\right)=\dfrac{mg}{c}\lim\limits_{t\to\infty}\left(1-e^{-ct/m}\right)=\dfrac{mg}{c}(1-0)$ [because $-ct/m\to-\infty$ as $t\to\infty$]

$=\dfrac{mg}{c}$, which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim\limits_{c\to0^+}v=\lim\limits_{c\to0^+}\dfrac{mg}{c}(1-e^{-ct/m})=mg\lim\limits_{c\to0^+}\dfrac{1-e^{-ct/m}}{c}$ [form is $\frac{0}{0}$]

$\overset{\text{H}}{=}mg\lim\limits_{c\to0^+}\dfrac{(-e^{-ct/m})\cdot(-t/m)}{1}=\dfrac{mgt}{m}\lim\limits_{c\to0^+}e^{-ct/m}=gt(1)=gt$

The velocity of a falling object in a vacuum is directly proportional to the amount of time it falls.

81. First we will find $\lim\limits_{n\to\infty}\left(1+\dfrac{r}{n}\right)^{nt}$, which is of the form 1^∞. $y=\left(1+\dfrac{r}{n}\right)^{nt} \;\Rightarrow\; \ln y=nt\ln\left(1+\dfrac{r}{n}\right)$, so

$\lim\limits_{n\to\infty}\ln y=\lim\limits_{n\to\infty}nt\ln\left(1+\dfrac{r}{n}\right)=t\lim\limits_{n\to\infty}\dfrac{\ln(1+r/n)}{1/n}\overset{\text{H}}{=}t\lim\limits_{n\to\infty}\dfrac{(-r/n^2)}{(1+r/n)(-1/n^2)}=t\lim\limits_{n\to\infty}\dfrac{r}{1+i/n}=tr \;\Rightarrow$

$\lim\limits_{n\to\infty}y=e^{rt}$. Thus, as $n\to\infty$, $A=A_0\left(1+\dfrac{r}{n}\right)^{nt}\to A_0e^{rt}$.

82. (a) $r=3,\ \rho=0.05 \;\Rightarrow\; P=\dfrac{1-10^{-\rho r^2}}{\rho r^2\ln10}=\dfrac{1-10^{-0.45}}{0.45\ln10}\approx0.62$, or about 62%.

(b) $r=2,\ \rho=0.05 \;\Rightarrow\; P=\dfrac{1-10^{-0.2}}{0.2\ln10}\approx0.80$, or about 80%.

Yes, it makes sense. Since measured brightness decreases with light entering farther from the center of the pupil, a smaller pupil radius means that the average brightness measurements are higher than when including light entering at larger radii.

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(c) $\lim_{r\to0^+} P = \lim_{r\to0^+} \dfrac{1-10^{-\rho r^2}}{\rho r^2 \ln 10} \overset{\text{H}}{=} \lim_{r\to0^+} \dfrac{-10^{-\rho r^2}(\ln 10)(-2\rho r)}{2\rho r(\ln 10)} = \lim_{r\to0^+} \dfrac{1}{10^{\rho r^2}} = 1$, or 100%.

We might expect that 100% of the brightness is sensed at the very center of the pupil, so a limit of 1 would make sense in this context if the radius r could approach 0. This result isn't physically possible because there are limitations on how small the pupil can shrink.

83. (a) $\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \dfrac{M}{1+Ae^{-kt}} = \dfrac{M}{1+A\cdot 0} = M$

It is to be expected that a population that is growing will eventually reach the maximum population size that can be supported.

(b) $\lim_{M\to\infty} P(t) = \lim_{M\to\infty} \dfrac{M}{1+\dfrac{M-P_0}{P_0}e^{-kt}} = \lim_{M\to\infty} \dfrac{M}{1+\left(\dfrac{M}{P_0}-1\right)e^{-kt}} \overset{\text{H}}{=} \lim_{M\to\infty} \dfrac{1}{\dfrac{1}{P_0}e^{-kt}} = P_0e^{kt}$

P_0e^{kt} is an exponential function.

84. (a) $\lim_{R\to r^+} v = \lim_{R\to r^+} \left[-c\left(\dfrac{r}{R}\right)^2 \ln\left(\dfrac{r}{R}\right)\right] = -cr^2 \lim_{R\to r^+} \left[\left(\dfrac{1}{R}\right)^2 \ln\left(\dfrac{r}{R}\right)\right] = -cr^2\cdot\dfrac{1}{r^2}\cdot\ln 1 = -c\cdot 0 = 0$

As the insulation of a metal cable becomes thinner, the velocity of an electrical impulse in the cable approaches zero.

(b) $\lim_{r\to0^+} v = \lim_{r\to0^+} \left[-c\left(\dfrac{r}{R}\right)^2 \ln\left(\dfrac{r}{R}\right)\right] = -\dfrac{c}{R^2} \lim_{r\to0^+} \left[r^2 \ln\left(\dfrac{r}{R}\right)\right]$ [form is $0\cdot\infty$]

$= -\dfrac{c}{R^2} \lim_{r\to0^+} \dfrac{\ln\left(\dfrac{r}{R}\right)}{\dfrac{1}{r^2}}$ [form is ∞/∞] $\overset{\text{H}}{=} -\dfrac{c}{R^2} \lim_{r\to0^+} \dfrac{\dfrac{R}{r}\cdot\dfrac{1}{R}}{\dfrac{-2}{r^3}} = -\dfrac{c}{R^2} \lim_{r\to0^+} \left(-\dfrac{r^2}{2}\right) = 0$

As the radius of the metal cable approaches zero, the velocity of an electrical impulse in the cable approaches zero.

85. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a} \frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} \overset{\text{H}}{=} \lim_{x\to a} \frac{\frac{1}{2}(2a^3x-x^4)^{-1/2}(2a^3-4x^3)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{\frac{1}{2}(2a^3a-a^4)^{-1/2}(2a^3-4a^3)-\frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}$$

$$= \frac{(a^4)^{-1/2}(-a^3)-\frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^3(a^4)^{-3/4}} = \frac{-a-\frac{1}{3}a}{-\frac{3}{4}} = \frac{4}{3}\left(\frac{4}{3}a\right) = \frac{16}{9}a$$

86. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ| = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$. So we have $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary

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trigonometry, $B(\theta) = \frac{1}{2}\,|QR|\,|PQ| = \frac{1}{2}(r - |OQ|)\,|PQ| = \frac{1}{2}(r - r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}\\ &= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}\\ &= \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta} = \lim_{\theta\to 0^+}\frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos 0} = \frac{1}{3}\end{aligned}$$

87. The limit, $L = \lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1+x}{x}\right)\right] = \lim_{x\to\infty}\left[x - x^2\ln\left(\frac{1}{x}+1\right)\right]$. Let $t = 1/x$, so as $x\to\infty$, $t\to 0^+$.

$$L = \lim_{t\to 0^+}\left[\frac{1}{t} - \frac{1}{t^2}\ln(t+1)\right] = \lim_{t\to 0^+}\frac{t-\ln(t+1)}{t^2} \overset{\text{H}}{=} \lim_{t\to 0^+}\frac{1-\dfrac{1}{t+1}}{2t} = \lim_{t\to 0^+}\frac{t/(t+1)}{2t} = \lim_{t\to 0^+}\frac{1}{2\,(t+1)} = \frac{1}{2}$$

Note: Starting the solution by factoring out x or x^2 leads to a more complicated solution.

88. $y = [f(x)]^{g(x)} \quad\Rightarrow\quad \ln y = g(x)\ln f(x)$. Since f is a positive function, $\ln f(x)$ is defined. Now $\lim_{x\to a}\ln y = \lim_{x\to a} g(x)\ln f(x) = -\infty$ since $\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} f(x) = 0 \quad\Rightarrow\quad \lim_{x\to a}\ln f(x) = -\infty$. Thus, if $t = \ln y$, $\lim_{x\to a} y = \lim_{t\to-\infty} e^t = 0$. Note that the limit, $\lim_{x\to a} g(x)\ln f(x)$, is *not* of the form $\infty\cdot 0$.

89. (a) We look for functions f and g whose individual limits are ∞ as $x\to 0$, but whose quotient has a limit of 7 as $x\to 0$.

One such pair of functions is $f(x) = \dfrac{7}{x^2}$ and $g(x) = \dfrac{1}{x^2}$. We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$, and

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{7/x^2}{1/x^2} = \lim_{x\to 0} 7 = 7.$$

(b) We look for functions f and g whose individual limits are ∞ as $x\to 0$, but whose difference has a limit of 7 as $x\to 0$.

One such pair of functions is $f(x) = \dfrac{1}{x^2} + 7$ and $g(x) = \dfrac{1}{x^2}$. We have $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = \infty$, and

$$\lim_{x\to 0}[f(x) - g(x)] = \lim_{x\to 0}\left[\left(\frac{1}{x^2}+7\right) - \frac{1}{x^2}\right] = \lim_{x\to 0} 7 = 7.$$

90. $L = \lim_{x\to 0}\left(\dfrac{\sin 2x}{x^3} + a + \dfrac{b}{x^2}\right) = \lim_{x\to 0}\dfrac{\sin 2x + ax^3 + bx}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{2\cos 2x + 3ax^2 + b}{3x^2}$. As $x\to 0$, $3x^2\to 0$, and $(2\cos 2x + 3ax^2 + b)\to b+2$, so the last limit exists only if $b+2=0$, that is, $b=-2$. Thus,

$\lim_{x\to 0}\dfrac{2\cos 2x + 3ax^2 - 2}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{-4\sin 2x + 6ax}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\dfrac{-8\cos 2x + 6a}{6} = \dfrac{6a-8}{6}$, which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b = -2$ and $a = \frac{4}{3}$.

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91. (a) We show that $\lim_{x\to 0}\dfrac{f(x)}{x^n}=0$ for every integer $n\ge 0$. Let $y=\dfrac{1}{x^2}$. Then

$$\lim_{x\to 0}\frac{f(x)}{x^{2n}}=\lim_{x\to 0}\frac{e^{-1/x^2}}{(x^2)^n}=\lim_{y\to\infty}\frac{y^n}{e^y}\overset{\text{H}}{=}\lim_{y\to\infty}\frac{ny^{n-1}}{e^y}\overset{\text{H}}{=}\cdots\overset{\text{H}}{=}\lim_{y\to\infty}\frac{n!}{e^y}=0 \quad\Rightarrow$$

$$\lim_{x\to 0}\frac{f(x)}{x^n}=\lim_{x\to 0}x^n\frac{f(x)}{x^{2n}}=\lim_{x\to 0}x^n\lim_{x\to 0}\frac{f(x)}{x^{2n}}=0. \text{ Thus, } f'(0)=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to 0}\frac{f(x)}{x}=0.$$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x\ne 0$. In fact, we prove by induction that for each $n\ge 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x)=p_n(x)f(x)/x^{k_n}$ for $x\ne 0$. This is true for $n=0$; suppose it is true for the nth derivative. Then $f'(x)=f(x)(2/x^3)$, so

$$\begin{aligned}f^{(n+1)}(x)&=\left[x^{k_n}\left[p_n'(x)\,f(x)+p_n(x)\,f'(x)\right]-k_nx^{k_n-1}p_n(x)\,f(x)\right]x^{-2k_n}\\&=\left[x^{k_n}p_n'(x)+p_n(x)(2/x^3)-k_nx^{k_n-1}p_n(x)\right]f(x)x^{-2k_n}\\&=\left[x^{k_n+3}p_n'(x)+2p_n(x)-k_nx^{k_n+2}\,p_n(x)\right]f(x)x^{-(2k_n+3)}\end{aligned}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0)=0$ for all n. By part (a), $f'(0)=0$. Suppose that $f^{(n)}(0)=0$. Then

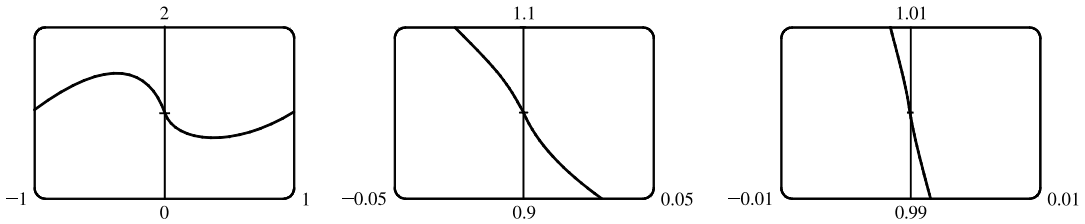
$$\begin{aligned}f^{(n+1)}(0)&=\lim_{x\to 0}\frac{f^{(n)}(x)-f^{(n)}(0)}{x-0}=\lim_{x\to 0}\frac{f^{(n)}(x)}{x}=\lim_{x\to 0}\frac{p_n(x)\,f(x)/x^{k_n}}{x}=\lim_{x\to 0}\frac{p_n(x)\,f(x)}{x^{k_n+1}}\\&=\lim_{x\to 0}p_n(x)\lim_{x\to 0}\frac{f(x)}{x^{k_n+1}}=p_n(0)\cdot 0=0\end{aligned}$$

92. (a) For f to be continuous, we need $\lim_{x\to 0}f(x)=f(0)=1$. We note that for $x\ne 0$, $\ln f(x)=\ln|x|^x=x\ln|x|$.

So $\lim_{x\to 0}\ln f(x)=\lim_{x\to 0}x\ln|x|=\lim_{x\to 0}\dfrac{\ln|x|}{1/x}\overset{\text{H}}{=}\lim_{x\to 0}\dfrac{1/x}{-1/x^2}=0$. Therefore, $\lim_{x\to 0}f(x)=\lim_{x\to 0}e^{\ln f(x)}=e^0=1$.

So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.



(c) To find f', we use logarithmic differentiation: $\ln f(x)=x\ln|x| \quad\Rightarrow\quad \dfrac{f'(x)}{f(x)}=x\left(\dfrac{1}{x}\right)+\ln|x| \quad\Rightarrow$

$f'(x)=f(x)(1+\ln|x|)=|x|^x(1+\ln|x|)$, $x\ne 0$. Now $f'(x)\to-\infty$ as $x\to 0$ [since $|x|^x\to 1$ and $(1+\ln|x|)\to-\infty$], so the curve has a vertical tangent at $(0,1)$ and is therefore not differentiable there.

The fact cannot be seen in the graphs in part (b) because $\ln|x|\to-\infty$ very slowly as $x\to 0$.

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4.5 Summary of Curve Sketching

1. $y = f(x) = x^3 + 3x^2 = x^2(x+3)$ **A.** f is a polynomial, so $D = \mathbb{R}$.

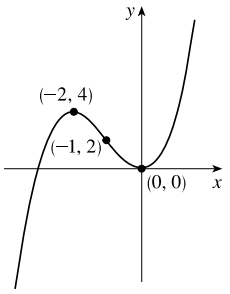
B. y-intercept $= f(0) = 0$, x-intercepts are 0 and -3 **C.** No symmetry

D. No asympote **E.** $f'(x) = 3x^2 + 6x = 3x(x+2) > 0 \quad \Leftrightarrow \quad x < -2$ or $x > 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$, and decreasing on $(-2, 0)$.

F. Local maximum value $f(-2) = 4$, local minimum value $f(0) = 0$

G. $f''(x) = 6x + 6 = 6(x+1) > 0 \quad \Leftrightarrow \quad x > -1$, so f is CU on $(-1, \infty)$ and CD on $(-\infty, -1)$. IP at $(-1, 2)$

H.



2. $y = f(x) = 2x^3 - 12x^2 + 18x = 2x(x^2 - 6x + 9) = 2x(x-3)^2$

A. $D = \mathbb{R}$ **B.** x-intercepts are 0 and 3, y-intercept $f(0) = 0$ **C.** No symmetry

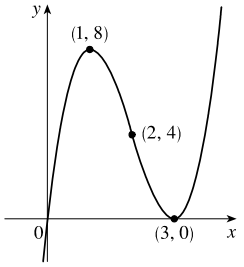
D. No asymptote

E. $f'(x) = 6x^2 - 24x + 18 = 6(x^2 - 4x + 3)$

$= 6(x-1)(x-3) > 0 \quad \Leftrightarrow \quad x < 1$ or $x > 3$

and $f'(x) < 0 \quad \Leftrightarrow \quad 1 < x < 3$, so f is increasing on $(-\infty, 1)$ and $(3, \infty)$, and decreasing on $(1, 3)$. **F.** Local maximum value $f(1) = 8$, local minimum value $f(3) = 0$ **G.** $f''(x) = 12x - 24 = 12(x-2) > 0 \quad \Leftrightarrow \quad x > 2$, so f is CU on $(2, \infty)$, and f is CD on $(-\infty, 2)$. IP at $(2, 4)$

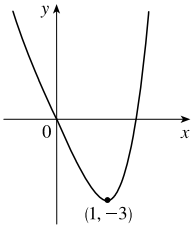
H.



3. $y = f(x) = x^4 - 4x = x(x^3 - 4)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and $\sqrt[3]{4}$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2 + x + 1) > 0 \quad \Leftrightarrow \quad x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** Local minimum value $f(1) = -3$, no local maximum **G.** $f''(x) = 12x^2 > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

H.



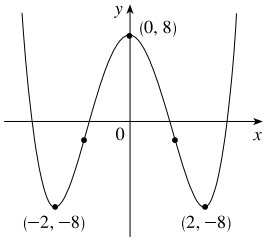
4. $y = f(x) = x^4 - 8x^2 + 8$ **A.** $D = \mathbb{R}$ **B.** y-intercept $f(0) = 8$; x-intercepts: $f(x) = 0 \quad \Rightarrow$ [by the quadratic formula] $x = \pm\sqrt{4 \pm 2\sqrt{2}} \approx \pm 2.61, \pm 1.08$ **C.** $f(-x) = f(x)$, so f is even and symmetric about the y-axis **D.** No asymptote

E. $f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2) > 0 \quad \Leftrightarrow \quad -2 < x < 0$ or $x > 2$, so f is increasing on $(-2, 0)$ and $(2, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(0, 2)$.

F. Local maximum value $f(0) = 8$, local minimum values $f(\pm 2) = -8$

G. $f''(x) = 12x^2 - 16 = 4(3x^2 - 4) > 0 \quad \Rightarrow \quad |x| > 2/\sqrt{3}$ [≈ 1.15], so f is CU on $\left(-\infty, -2/\sqrt{3}\right)$ and $\left(2/\sqrt{3}, \infty\right)$, and f is CD on $\left(-2/\sqrt{3}, 2/\sqrt{3}\right)$. IP at $\left(\pm 2/\sqrt{3}, -\frac{8}{9}\right)$

H.



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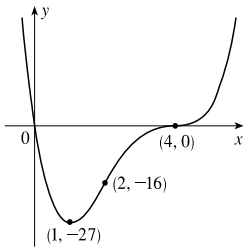
5. $y = f(x) = x(x-4)^3$ **A.** $D = \mathbb{R}$ **B.** x-intercepts are 0 and 4, y-intercept $f(0) = 0$ **C.** No symmetry

D. No asymptote **H.**



E. $f'(x) = x \cdot 3(x-4)^2 + (x-4)^3 \cdot 1 = (x-4)^2[3x + (x-4)]$

$= (x-4)^2(4x-4) = 4(x-1)(x-4)^2 > 0 \quad \Leftrightarrow$

$x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$.

F. Local minimum value $f(1) = -27$, no local maximum value

G. $f''(x) = 4[(x-1) \cdot 2(x-4) + (x-4)^2 \cdot 1] = 4(x-4)[2(x-1) + (x-4)]$

$= 4(x-4)(3x-6) = 12(x-4)(x-2) < 0 \quad \Leftrightarrow$

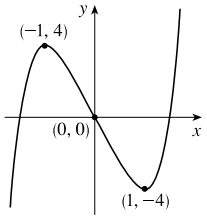
$2 < x < 4$, so f is CD on $(2, 4)$ and CU on $(-\infty, 2)$ and $(4, \infty)$. IPs at $(2, -16)$ and $(4, 0)$

6. $y = f(x) = x^5 - 5x = x(x^4 - 5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts $\pm\sqrt[4]{5}$ and 0, y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1)$ **H.**

$= 5(x+1)(x-1)(x^2+1) > 0 \quad \Leftrightarrow$

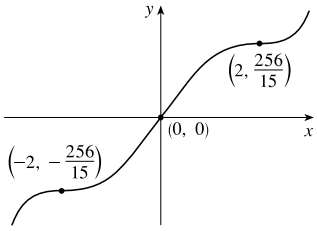


$x < -1$ or $x > 1$, so f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and f is decreasing on $(-1, 1)$. **F.** Local maximum value $f(-1) = 4$, local minimum value $f(1) = -4$ **G.** $f''(x) = 20x^3 > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. IP at $(0, 0)$

7. $y = f(x) = \frac{1}{5}x^5 - \frac{8}{3}x^3 + 16x = x\left(\frac{1}{5}x^4 - \frac{8}{3}x^2 + 16\right)$ **A.** $D = \mathbb{R}$ **B.** x-intercept 0, y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd; the curve is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x^4 - 8x^2 + 16 = (x^2 - 4)^2 = (x+2)^2(x-2)^2 > 0$ for all x except ± 2, so f is increasing on $\mathbb{R}$. **F.** There is no local maximum or minimum value. **H.**



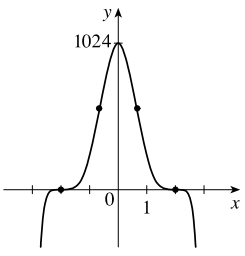
G. $f''(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2) > 0 \quad \Leftrightarrow$

$-2 < x < 0$ or $x > 2$, so f is CU on $(-2, 0)$ and $(2, \infty)$, and f is CD on $(-\infty, -2)$ and $(0, 2)$. IP at $\left(-2, -\frac{256}{15}\right)$, $(0, 0)$, and $\left(2, \frac{256}{15}\right)$

8. $y = f(x) = (4 - x^2)^5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 4^5 = 1024$; x-intercepts: ± 2 **C.** $f(-x) = f(x) \quad \Rightarrow$ f is even; the curve is symmetric about the y-axis. **D.** No asymptote **E.** $f'(x) = 5(4 - x^2)^4(-2x) = -10x(4 - x^2)^4$, so for $x \neq \pm 2$ we have $f'(x) > 0 \quad \Leftrightarrow \quad x < 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x > 0$. Thus, f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. **F.** Local maximum value $f(0) = 1024$

G. $f''(x) = -10x \cdot 4(4 - x^2)^3(-2x) + (4 - x^2)^4(-10)$ **H.**

$= -10(4 - x^2)^3[-8x^2 + 4 - x^2] = -10(4 - x^2)^3(4 - 9x^2)$



so $f''(x) = 0 \quad \Leftrightarrow \quad x = \pm 2, \pm\frac{2}{3}$. $f''(x) > 0 \quad \Leftrightarrow \quad -2 < x < -\frac{2}{3}$ and $\frac{2}{3} < x < 2$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -2, -\frac{2}{3} < x < \frac{2}{3}$, and $x > 2$, so f is CU on $(-\infty, 2)$, $\left(-\frac{2}{3}, \frac{2}{3}\right)$, and $(2, \infty)$, and CD on $\left(-2, -\frac{2}{3}\right)$ and $\left(\frac{2}{3}, 2\right)$. IP at $(\pm 2, 0)$ and $\left(\pm\frac{2}{3}, \left(\frac{32}{9}\right)^5\right) \approx (\pm 0.67, 568.25)$

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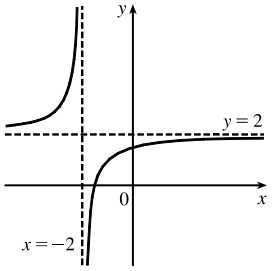
9. $y = f(x) = \dfrac{2x+3}{x+2}$ **A.** $D = \{x \mid x \neq -2\} = (-\infty,-2) \cup (-2,\infty)$ **B.** x-intercept $= -\frac{3}{2}$, y-intercept $= f(0) = \frac{3}{2}$

C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{2x+3}{x+2} = 2$, so $y = 2$ is a HA. $\lim\limits_{x\to -2^-} \dfrac{2x+3}{x+2} = \infty$, $\lim\limits_{x\to -2^+} \dfrac{2x+3}{x+2} = -\infty$, so $x = -2$ is a VA. **E.** $f'(x) = \dfrac{(x+2)\cdot 2 - (2x+3)\cdot 1}{(x+2)^2} = \dfrac{1}{(x+2)^2} > 0$ for $x \neq -2$, so f is increasing on $(-\infty,-2)$ and $(-2,\infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{-2}{(x+2)^3} > 0 \quad \Leftrightarrow \quad x < -2$, so f is CU on $(-\infty,-2)$ and CD on $(-2,\infty)$. No IP

H.



10. $y = f(x) = \dfrac{x^2+5x}{25-x^2} = \dfrac{x(x+5)}{(5+x)(5-x)} = \dfrac{x}{5-x}$ for $x \neq -5$. There is a hole in the graph at $\left(-5,-\frac{1}{2}\right)$.

A. $D = \{x \mid x \neq \pm 5\} = (-\infty,-5) \cup (-5,5) \cup (5,\infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

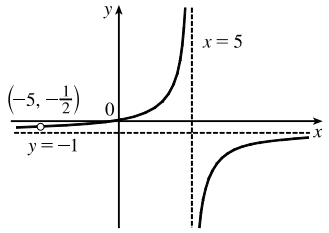
D. $\lim\limits_{x\to\pm\infty} \dfrac{x}{5-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x\to 5^-} \dfrac{x}{5-x} = \infty$, $\lim\limits_{x\to 5^+} \dfrac{x}{5-x} = -\infty$, so $x = 5$ is a VA.

E. $f'(x) = \dfrac{(5-x)(1) - x(-1)}{(5-x)^2} = \dfrac{5}{(5-x)^2} > 0$ for all x in D, so f is increasing on $(-\infty,-5)$, $(-5,5)$, and $(5,\infty)$. **F.** No extrema

G. $f'(x) = 5(5-x)^{-2} \quad \Rightarrow$

$f''(x) = -10(5-x)^{-3}(-1) = \dfrac{10}{(5-x)^3} > 0 \quad \Leftrightarrow \quad x < 5$, so f is CU on $(-\infty,-5)$ and $(-5,5)$, and f is CD on $(5,\infty)$. No IP

H.



11. $y = f(x) = \dfrac{x-x^2}{2-3x+x^2} = \dfrac{x(1-x)}{(1-x)(2-x)} = \dfrac{x}{2-x}$ for $x \neq 1$. There is a hole in the graph at $(1,1)$.

A. $D = \{x \mid x \neq 1,2\} = (-\infty,1) \cup (1,2) \cup (2,\infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry

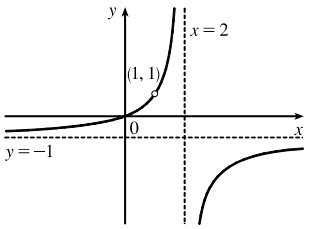
D. $\lim\limits_{x\to\pm\infty} \dfrac{x}{2-x} = -1$, so $y = -1$ is a HA. $\lim\limits_{x\to 2^-} \dfrac{x}{2-x} = \infty$, $\lim\limits_{x\to 2^+} \dfrac{x}{2-x} = -\infty$, so $x = 2$ is a VA.

E. $f'(x) = \dfrac{(2-x)(1) - x(-1)}{(2-x)^2} = \dfrac{2}{(2-x)^2} > 0$ $[x \neq 1,2]$, so f is increasing on $(-\infty,1)$, $(1,2)$, and $(2,\infty)$. **F.** No extrema

G. $f'(x) = 2(2-x)^{-2} \quad \Rightarrow$

$f''(x) = -4(2-x)^{-3}(-1) = \dfrac{4}{(2-x)^3} > 0 \quad \Leftrightarrow \quad x < 2$, so f is CU on $(-\infty,1)$ and $(1,2)$, and f is CD on $(2,\infty)$. No IP

H.



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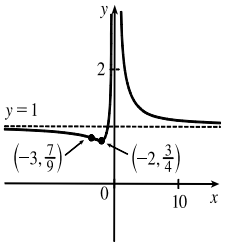
12. $y = f(x) = 1 + \dfrac{1}{x} + \dfrac{1}{x^2} = \dfrac{x^2 + x + 1}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** y-intercept: none $[x \neq 0]$; x-intercepts: $f(x) = 0 \Leftrightarrow x^2 + x + 1 = 0$, there is no real solution, and hence, no x-intercept **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} \left(1 + \dfrac{1}{x} + \dfrac{1}{x^2}\right) = 1$, so $y = 1$ is a HA. $\lim\limits_{x \to 0} f(x) = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = -\dfrac{1}{x^2} - \dfrac{2}{x^3} = \dfrac{-x - 2}{x^3}$. $f'(x) > 0 \Leftrightarrow -2 < x < 0$ and $f'(x) < 0 \Leftrightarrow x < -2$ or $x > 0$, so f is increasing on $(-2, 0)$ and decreasing on $(-\infty, -2)$ and $(0, \infty)$. **F.** Local minimum value $f(-2) = \frac{3}{4}$; no local maximum **G.** $f''(x) = \dfrac{2}{x^3} + \dfrac{6}{x^4} = \dfrac{2x + 6}{x^4}$. $f''(x) < 0 \Leftrightarrow x < -3$ and $f''(x) > 0 \Leftrightarrow -3 < x < 0$ and $x > 0$, so f is CD on $(-\infty, -3)$ and CU on $(-3, 0)$ and $(0, \infty)$. IP at $\left(-3, \frac{7}{9}\right)$

H.



13. $y = f(x) = \dfrac{x}{x^2 - 4} = \dfrac{x}{(x + 2)(x - 2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x \to 2^+} \dfrac{x}{x^2 - 4} = \infty$, $\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to -2^+} f(x) = \infty$, $\lim\limits_{x \to -2^-} f(x) = -\infty$, so $x = \pm 2$ are VAs. $\lim\limits_{x \to \pm\infty} \dfrac{x}{x^2 - 4} = 0$, so $y = 0$ is a HA. **E.** $f'(x) = \dfrac{(x^2 - 4)(1) - x(2x)}{(x^2 - 4)^2} = -\dfrac{x^2 + 4}{(x^2 - 4)^2} < 0$ for all x in D, so f is decreasing on $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$.

F. No local extrema

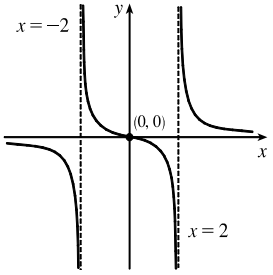
G. $f''(x) = -\dfrac{(x^2 - 4)^2(2x) - (x^2 + 4)2(x^2 - 4)(2x)}{[(x^2 - 4)^2]^2}$

$= -\dfrac{2x(x^2 - 4)[(x^2 - 4) - 2(x^2 + 4)]}{(x^2 - 4)^4}$

$= -\dfrac{2x(-x^2 - 12)}{(x^2 - 4)^3} = \dfrac{2x(x^2 + 12)}{(x + 2)^3(x - 2)^3}$.

$f''(x) < 0$ if $x < -2$ or $0 < x < 2$, so f is CD on $(-\infty, -2)$ and $(0, 2)$, and CU on $(-2, 0)$ and $(2, \infty)$. IP at $(0, 0)$

H.



14. $y = f(x) = \dfrac{1}{x^2 - 4} = \dfrac{1}{(x + 2)(x - 2)}$ **A.** $D = (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$ **B.** No x-intercept, y-intercept $= f(0) = -\frac{1}{4}$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim\limits_{x \to 2^+} \dfrac{1}{x^2 - 4} = \infty$, $\lim\limits_{x \to 2^-} f(x) = -\infty$, $\lim\limits_{x \to -2^+} f(x) = -\infty$, $\lim\limits_{x \to -2^-} f(x) = \infty$, so $x = \pm 2$ are VAs. $\lim\limits_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a HA. **E.** $f'(x) = -\dfrac{2x}{(x^2 - 4)^2}$ [Reciprocal Rule] > 0 if $x < 0$ and x is in D, so f is increasing on $(-\infty, -2)$ and $(-2, 0)$. f is decreasing on $(0, 2)$ and $(2, \infty)$. **F.** Local maximum value $f(0) = -\frac{1}{4}$, no local minimum value

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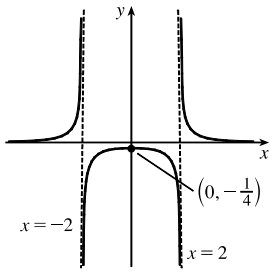
G. $f''(x) = \dfrac{(x^2-4)^2(-2) - (-2x)2(x^2-4)(2x)}{[(x^2-4)^2]^2}$

$= \dfrac{-2(x^2-4)\left[(x^2-4) - 4x^2\right]}{(x^2-4)^4}$

$= \dfrac{-2(-3x^2-4)}{(x^2-4)^3} = \dfrac{2(3x^2+4)}{(x^2-4)^3}$

$f''(x) < 0 \quad \Leftrightarrow \quad -2 < x < 2$, so f is CD on $(-2, 2)$ and CU on $(-\infty, -2)$ and $(2, \infty)$. No IP

H.



15. $y = f(x) = \dfrac{x^2}{x^2+3} = \dfrac{(x^2+3)-3}{x^2+3} = 1 - \dfrac{3}{x^2+3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ **C.** $f(-x) = f(x)$, so f is even; the graph is symmetric about the y-axis.

D. $\lim\limits_{x \to \pm\infty} \dfrac{x^2}{x^2+3} = 1$, so $y = 1$ is a HA. No VA. **E.** Using the Reciprocal Rule, $f'(x) = -3 \cdot \dfrac{-2x}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$.

$f'(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$, so f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

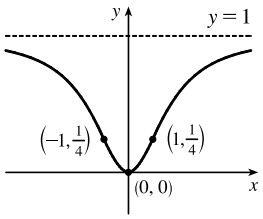
F. Local minimum value $f(0) = 0$, no local maximum.

G. $f''(x) = \dfrac{(x^2+3)^2 \cdot 6 - 6x \cdot 2(x^2+3) \cdot 2x}{[(x^2+3)^2]^2}$

$= \dfrac{6(x^2+3)[(x^2+3) - 4x^2]}{(x^2+3)^4} = \dfrac{6(3-3x^2)}{(x^2+3)^3} = \dfrac{-18(x+1)(x-1)}{(x^2+3)^3}$

$f''(x)$ is negative on $(-\infty, -1)$ and $(1, \infty)$ and positive on $(-1, 1)$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$ and CU on $(-1, 1)$. IP at $\left(\pm 1, \frac{1}{4}\right)$

H.



16. $y = f(x) = \dfrac{(x-1)^2}{x^2+1} \geq 0$ with equality $\quad \Leftrightarrow \quad x = 1$. **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 1$; x-intercept 1 **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} f(x) = \lim\limits_{x \to \pm\infty} \dfrac{x^2-2x+1}{x^2+1} = \lim\limits_{x \to \pm\infty} \dfrac{1 - 2/x + 1/x^2}{1 + 1/x^2} = 1$, so $y = 1$ is a HA. No VA

E. $f'(x) = \dfrac{(x^2+1)2(x-1) - (x-1)^2(2x)}{(x^2+1)^2} = \dfrac{2(x-1)\left[(x^2+1) - x(x-1)\right]}{(x^2+1)^2} = \dfrac{2(x-1)(x+1)}{(x^2+1)^2} < 0 \quad \Leftrightarrow$

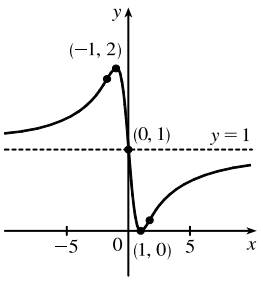
$-1 < x < 1$, so f is decreasing on $(-1, 1)$ and increasing on $(-\infty, -1)$ and $(1, \infty)$ **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = 0$

G. $f''(x) = \dfrac{(x^2+1)^2(4x) - (2x^2-2)2(x^2+1)(2x)}{[(x^2+1)^2]^2} = \dfrac{4x(x^2+1)\left[(x^2+1) - (2x^2-2)\right]}{(x^2+1)^4} = \dfrac{4x(3-x^2)}{(x^2+1)^3}$.

$f''(x) > 0 \quad \Leftrightarrow \quad x < -\sqrt{3}$ or $0 < x < \sqrt{3}$, so f is CU on $\left(-\infty, -\sqrt{3}\right)$ and $\left(0, \sqrt{3}\right)$, and f is CD on $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, \infty\right)$.

$f\left(\pm\sqrt{3}\right) = \frac{1}{4}\left(\sqrt{3} \mp 1\right)^2 = \frac{1}{4}\left(4 \mp 2\sqrt{3}\right) = 1 \mp \frac{1}{2}\sqrt{3}\,[\approx 0.13, 1.87]$, so there are IPs at $\left(-\sqrt{3}, 1 + \frac{1}{2}\sqrt{3}\right)$, $(0, 1)$, and $\left(\sqrt{3}, 1 - \frac{1}{2}\sqrt{3}\right)$. Note that the graph is symmetric about the point $(0, 1)$.

H.



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17. $y = f(x) = \dfrac{x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \Leftrightarrow x = 1$

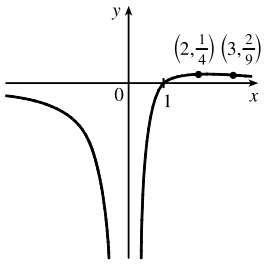
C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x-1}{x^2} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 0} \dfrac{x-1}{x^2} = -\infty$, so $x = 0$ is a VA.

E. $f'(x) = \dfrac{x^2 \cdot 1 - (x-1)\cdot 2x}{(x^2)^2} = \dfrac{-x^2+2x}{x^4} = \dfrac{-(x-2)}{x^3}$, so $f'(x) > 0 \Leftrightarrow 0 < x < 2$ and $f'(x) < 0 \Leftrightarrow$ $x < 0$ or $x > 2$. Thus, f is increasing on $(0, 2)$ and decreasing on $(-\infty, 0)$ and $(2, \infty)$. **F.** No local minimum, local maximum value $f(2) = \frac{1}{4}$.

G. $f''(x) = \dfrac{x^3 \cdot (-1) - [-(x-2)]\cdot 3x^2}{(x^3)^2} = \dfrac{2x^3 - 6x^2}{x^6} = \dfrac{2(x-3)}{x^4}$.

$f''(x)$ is negative on $(-\infty, 0)$ and $(0, 3)$ and positive on $(3, \infty)$, so f is CD on $(-\infty, 0)$ and $(0, 3)$ and CU on $(3, \infty)$. IP at $\left(3, \frac{2}{9}\right)$

H.



18. $y = f(x) = \dfrac{x}{x^3-1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$

C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x}{x^3-1} = 0$, so $y = 0$ is a HA. $\lim\limits_{x\to 1^-} f(x) = -\infty$ and $\lim\limits_{x\to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

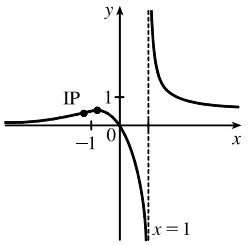
E. $f'(x) = \dfrac{(x^3-1)(1) - x(3x^2)}{(x^3-1)^2} = \dfrac{-2x^3-1}{(x^3-1)^2}$. $f'(x) = 0 \Rightarrow x = -\sqrt[3]{1/2}$. $f'(x) > 0 \Leftrightarrow x < -\sqrt[3]{1/2}$ and $f'(x) < 0 \Leftrightarrow -\sqrt[3]{1/2} < x < 1$ and $x > 1$, so f is increasing on $\left(-\infty, -\sqrt[3]{1/2}\right)$ and decreasing on $\left(-\sqrt[3]{1/2}, 1\right)$ and $(1, \infty)$. **F.** Local maximum value $f\left(-\sqrt[3]{1/2}\right) = \frac{2}{3}\sqrt[3]{1/2}$; no local minimum

G. $f''(x) = \dfrac{(x^3-1)^2(-6x^2) - (-2x^3-1)2(x^3-1)(3x^2)}{[(x^3-1)^2]^2}$

$= \dfrac{-6x^2(x^3-1)[(x^3-1) - (2x^3+1)]}{(x^3-1)^4} = \dfrac{6x^2(x^3+2)}{(x^3-1)^3}$.

$f''(x) > 0 \Leftrightarrow x < -\sqrt[3]{2}$ and $x > 1$, $f''(x) < 0 \Leftrightarrow -\sqrt[3]{2} < x < 0$ and $0 < x < 1$, so f is CU on $\left(-\infty, -\sqrt[3]{2}\right)$ and $(1, \infty)$ and CD on $\left(-\sqrt[3]{2}, 1\right)$. IP at $\left(-\sqrt[3]{2}, \frac{1}{3}\sqrt[3]{2}\right)$

H.



19. $y = f(x) = \dfrac{x^3}{x^3+1} = \dfrac{x^3}{(x+1)(x^2-x+1)}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: $f(x) = 0 \Leftrightarrow x = 0$ **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} \dfrac{x^3}{x^3+1} = \dfrac{1}{1+1/x^3} = 1$, so $y = 1$ is a HA. $\lim\limits_{x\to -1^-} f(x) = \infty$ and $\lim\limits_{x\to -1^+} f(x) = -\infty$, so $x = -1$ is a VA. **E.** $f'(x) = \dfrac{(x^3+1)(3x^2) - x^3(3x^2)}{(x^3+1)^2} = \dfrac{3x^2}{(x^3+1)^2}$. $f'(x) > 0$ for $x \neq -1$ (not in the domain) and $x \neq 0$ ($f' = 0$), so f is increasing on $(-\infty, -1)$, $(-1, 0)$, and $(0, \infty)$, and furthermore, by Exercise 4.3.97, f is increasing on $(-\infty, -1)$, and $(-1, \infty)$. **F.** No local extrema

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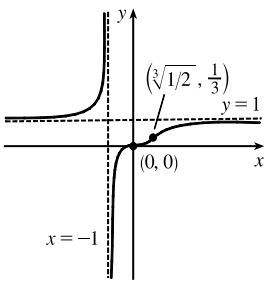
G. $f''(x) = \dfrac{(x^3+1)^2(6x) - 3x^2[2(x^3+1)(3x^2)]}{[(x^3+1)^2]^2}$

$= \dfrac{(x^3+1)(6x)[(x^3+1) - 3x^3]}{(x^3+1)^4} = \dfrac{6x(1-2x^3)}{(x^3+1)^3}$

$f''(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $0 < x < \sqrt[3]{\frac{1}{2}}$ $[\approx 0.79]$, so f is CU on $(-\infty, -1)$ and $\left(0, \sqrt[3]{\frac{1}{2}}\right)$ and CD on $(-1, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$. There are IPs at $(0, 0)$ and $\left(\sqrt[3]{\frac{1}{2}}, \frac{1}{3}\right)$.

H.



20. $y = f(x) = \dfrac{x^3}{x-2} = x^2 + 2x + 4 + \dfrac{8}{x-2}$ [by long division] **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercept $= 0$, y-intercept $= f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to 2^-} \dfrac{x^3}{x-2} = -\infty$ and $\lim\limits_{x \to 2^+} \dfrac{x^3}{x-2} = \infty$, so $x = 2$ is a VA.

There are no horizontal or slant asymptotes. *Note:* Since $\lim\limits_{x \to \pm\infty} \dfrac{8}{x-2} = 0$, the parabola $y = x^2 + 2x + 4$ is approached asymptotically as $x \to \pm\infty$.

E. $f'(x) = \dfrac{(x-2)(3x^2) - x^3(1)}{(x-2)^2} = \dfrac{x^2[3(x-2) - x]}{(x-2)^2} = \dfrac{x^2(2x-6)}{(x-2)^2} = \dfrac{2x^2(x-3)}{(x-2)^2} > 0 \quad \Leftrightarrow \quad x > 3$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$ or $0 < x < 2$ or $2 < x < 3$, so f is increasing on $(3, \infty)$ and f is decreasing on $(-\infty, 2)$ and $(2, 3)$.

F. Local minimum value $f(3) = 27$, no local maximum value **G.** $f'(x) = 2\dfrac{x^3 - 3x^2}{(x-2)^2} \quad \Rightarrow$

$f''(x) = 2\dfrac{(x-2)^2(3x^2-6x) - (x^3-3x^2)2(x-2)}{[(x-2)^2]^2}$

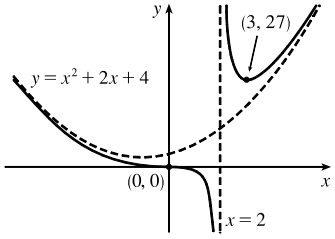
$= 2\dfrac{(x-2)x[(x-2)(3x-6) - (x^2-3x)2]}{(x-2)^4}$

$= \dfrac{2x(3x^2 - 12x + 12 - 2x^2 + 6x)}{(x-2)^3}$

$= \dfrac{2x(x^2 - 6x + 12)}{(x-2)^3} > 0 \quad \Leftrightarrow$

$x < 0$ or $x > 2$, so f is CU on $(-\infty, 0)$ and $(2, \infty)$, and f is CD on $(0, 2)$. IP at $(0, 0)$

H.

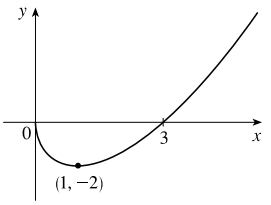


21. $y = f(x) = (x-3)\sqrt{x} = x^{3/2} - 3x^{1/2}$ **A.** $D = [0, \infty)$ **B.** x-intercepts: $0, 3$; y-intercept $= f(0) = 0$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{3}{2}x^{1/2} - \frac{3}{2}x^{-1/2} = \frac{3}{2}x^{-1/2}(x-1) = \dfrac{3(x-1)}{2\sqrt{x}} > 0 \quad \Leftrightarrow \quad x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(0, 1)$.

F. Local minimum value $f(1) = -2$, no local maximum value

G. $f''(x) = \frac{3}{4}x^{-1/2} + \frac{3}{4}x^{-3/2} = \frac{3}{4}x^{-3/2}(x+1) = \dfrac{3(x+1)}{4x^{3/2}} > 0$ for $x > 0$, so f is CU on $(0, \infty)$. No IP

H.



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22. $y = f(x) = (x-4)\sqrt[3]{x} = x^{4/3} - 4x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 0$; x-intercepts: 0 and 4

C. No symmetry **D.** No asymptote

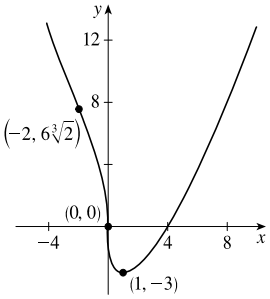
E. $f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-2/3}(x-1) = \dfrac{4(x-1)}{3x^{2/3}}$. $f'(x) > 0 \Leftrightarrow$ $x > 1$, so f is increasing on $(1,\infty)$ and f is decreasing on $(-\infty, 1)$.

F. Local minimum value $f(1) = -3$

G. $f''(x) = \frac{4}{9}x^{-2/3} + \frac{8}{9}x^{-5/3} = \frac{4}{9}x^{-5/3}(x+2) = \dfrac{4(x+2)}{9x^{5/3}}$.

$f''(x) < 0 \Leftrightarrow -2 < x < 0$, so f is CD on $(-2, 0)$, and f is CU on $(-\infty, -2)$ and $(0,\infty)$. There are IPs at $\left(-2, 6\sqrt[3]{2}\right)$ and $(0,0)$.

H.



23. $y = f(x) = \sqrt{x^2 + x - 2} = \sqrt{(x+2)(x-1)}$ **A.** $D = \{x \mid (x+2)(x-1) \geq 0\} = (-\infty, -2] \cup [1,\infty)$

B. y-intercept: none; x-intercepts: -2 and 1 **C.** No symmetry **D.** No asymptote

E. $f'(x) = \frac{1}{2}(x^2 + x - 2)^{-1/2}(2x+1) = \dfrac{2x+1}{2\sqrt{x^2+x-2}}$, $f'(x) = 0$ if $x = -\frac{1}{2}$, but $-\frac{1}{2}$ is not in the domain.

$f'(x) > 0 \Rightarrow x > -\frac{1}{2}$ and $f'(x) < 0 \Rightarrow x < -\frac{1}{2}$, so (considering the domain) f is increasing on $(1,\infty)$ and f is decreasing on $(-\infty, -2)$. **F.** No local extrema

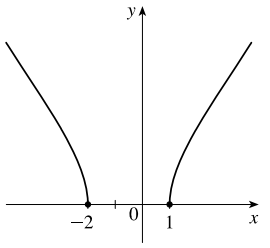
G. $f''(x) = \dfrac{2(x^2+x-2)^{1/2}(2) - (2x+1)\cdot 2\cdot \frac{1}{2}(x^2+x-2)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x-2}\right)^2}$

$= \dfrac{(x^2+x-2)^{-1/2}\left[4(x^2+x-2) - (4x^2+4x+1)\right]}{4(x^2+x-2)}$

$= \dfrac{-9}{4(x^2+x-2)^{3/2}} < 0$

so f is CD on $(-\infty, -2)$ and $(1,\infty)$. No IP

H.



24. $y = f(x) = \sqrt{x^2 + x} - x = \sqrt{x(x+1)} - x$ **A.** $D = (-\infty, -1] \cup [0,\infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow \sqrt{x^2+x} = x \Rightarrow x^2 + x = x^2 \Rightarrow x = 0$ **C.** No symmetry

D. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \left(\sqrt{x^2+x} - x\right)\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim\limits_{x\to\infty} \dfrac{x^2+x-x^2}{\sqrt{x^2+x}+x}$

$= \lim\limits_{x\to\infty} \dfrac{x/x}{\left(\sqrt{x^2+x}+x\right)/x} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+1/x}+1} = \dfrac{1}{2}$, so $y = \frac{1}{2}$ is a HA. No VA

E. $f'(x) = \frac{1}{2}(x^2+x)^{-1/2}(2x+1) - 1 = \dfrac{2x+1}{2\sqrt{x^2+x}} - 1 > 0 \Leftrightarrow 2x+1 > 2\sqrt{x^2+x} \Leftrightarrow$

$x + \frac{1}{2} > \sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}$. Keep in mind that the domain excludes the interval $(-1, 0)$. When $x + \frac{1}{2}$ is positive (for $x \geq 0$), the last inequality is *true* since the value of the radical is less than $x + \frac{1}{2}$. When $x + \frac{1}{2}$ is negative (for $x \leq -1$), the last inequality is *false* since the value of the radical is positive. So f is increasing on $(0,\infty)$ and decreasing on $(-\infty, -1)$.

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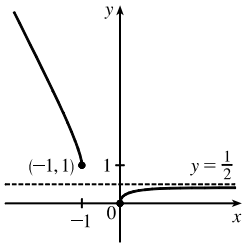
F. No local extrema

G. $f''(x) = \dfrac{2(x^2+x)^{1/2}(2) - (2x+1)\cdot 2\cdot \frac{1}{2}(x^2+x)^{-1/2}(2x+1)}{\left(2\sqrt{x^2+x}\right)^2}$

$= \dfrac{(x^2+x)^{-1/2}[4(x^2+x) - (2x+1)^2]}{4(x^2+x)} = \dfrac{-1}{4(x^2+x)^{3/2}}.$

$f''(x) < 0$ when it is defined, so f is CD on $(-\infty, -1)$ and $(0, \infty)$. No IP

H.



25. $y = f(x) = x/\sqrt{x^2+1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 0$

C. $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/x} = \lim\limits_{x\to\infty} \dfrac{x/x}{\sqrt{x^2+1}/\sqrt{x^2}} = \lim\limits_{x\to\infty} \dfrac{1}{\sqrt{1+1/x^2}} = \dfrac{1}{\sqrt{1+0}} = 1$

and

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2+1}} = \lim\limits_{x\to-\infty} \dfrac{x/x}{\sqrt{x^2+1}/x} = \lim\limits_{x\to-\infty} \dfrac{x/x}{\sqrt{x^2+1}/\left(-\sqrt{x^2}\right)} = \lim\limits_{x\to-\infty} \dfrac{1}{-\sqrt{1+1/x^2}}$

$= \dfrac{1}{-\sqrt{1+0}} = -1$ so $y = \pm 1$ are HA. No VA

E. $f'(x) = \dfrac{\sqrt{x^2+1} - x\cdot \dfrac{2x}{2\sqrt{x^2+1}}}{[(x^2+1)^{1/2}]^2} = \dfrac{x^2+1-x^2}{(x^2+1)^{3/2}} = \dfrac{1}{(x^2+1)^{3/2}} > 0$ for all x, so f is increasing on $\mathbb{R}$.

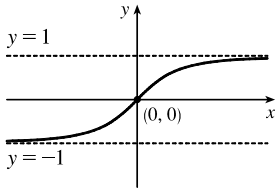
F. No extreme values

G. $f''(x) = -\frac{3}{2}(x^2+1)^{-5/2}\cdot 2x = \dfrac{-3x}{(x^2+1)^{5/2}}$, so $f''(x) > 0$ for $x < 0$

and $f''(x) < 0$ for $x > 0$. Thus, f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$.

IP at $(0, 0)$

H.



26. $y = f(x) = x\sqrt{2-x^2}$ **A.** $D = \left[-\sqrt{2}, \sqrt{2}\right]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow$

$x = 0, \pm\sqrt{2}$. **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote

E. $f'(x) = x\cdot \dfrac{-x}{\sqrt{2-x^2}} + \sqrt{2-x^2} = \dfrac{-x^2+2-x^2}{\sqrt{2-x^2}} = \dfrac{2(1+x)(1-x)}{\sqrt{2-x^2}}$. $f'(x)$ is negative for $-\sqrt{2} < x < -1$

and $1 < x < \sqrt{2}$, and positive for $-1 < x < 1$, so f is decreasing on $\left(-\sqrt{2}, -1\right)$ and $\left(1, \sqrt{2}\right)$ and increasing on $(-1, 1)$.

F. Local minimum value $f(-1) = -1$, local maximum value $f(1) = 1$.

G. $f''(x) = \dfrac{\sqrt{2-x^2}(-4x) - (2-2x^2)\dfrac{-x}{\sqrt{2-x^2}}}{[(2-x^2)^{1/2}]^2}$

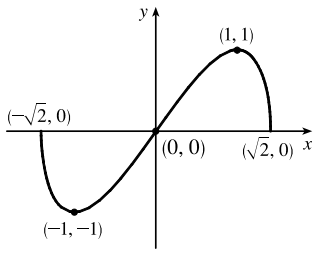
$= \dfrac{(2-x^2)(-4x) + (2-2x^2)x}{(2-x^2)^{3/2}} = \dfrac{2x^3-6x}{(2-x^2)^{3/2}} = \dfrac{2x(x^2-3)}{(2-x^2)^{3/2}}$

Since $x^2 - 3 < 0$ for x in $\left[-\sqrt{2}, \sqrt{2}\right]$, $f''(x) > 0$ for $-\sqrt{2} < x < 0$ and

$f''(x) < 0$ for $0 < x < \sqrt{2}$. Thus, f is CU on $\left(-\sqrt{2}, 0\right)$ and CD on $\left(0, \sqrt{2}\right)$.

The only IP is $(0, 0)$.

H.



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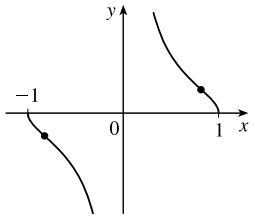
27. $y = f(x) = \sqrt{1-x^2}/x$ **A.** $D = \{x \mid |x| \le 1, x \ne 0\} = [-1,0) \cup (0,1]$ **B.** x-intercepts ± 1, no y-intercept

C. $f(-x) = -f(x)$, so the curve is symmetric about $(0,0)$. **D.** $\lim\limits_{x\to 0^+} \dfrac{\sqrt{1-x^2}}{x} = \infty$, $\lim\limits_{x\to 0^-} \dfrac{\sqrt{1-x^2}}{x} = -\infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{\left(-x^2/\sqrt{1-x^2}\right) - \sqrt{1-x^2}}{x^2} = -\dfrac{1}{x^2\sqrt{1-x^2}} < 0$, so f is decreasing on $(-1,0)$ and $(0,1)$. **F.** No extreme values

G. $f''(x) = \dfrac{2-3x^2}{x^3(1-x^2)^{3/2}} > 0 \quad \Leftrightarrow \quad -1 < x < -\sqrt{\frac{2}{3}}$ or $0 < x < \sqrt{\frac{2}{3}}$, so f is CU on $\left(-1, -\sqrt{\frac{2}{3}}\right)$ and $\left(0, \sqrt{\frac{2}{3}}\right)$ and CD on $\left(-\sqrt{\frac{2}{3}}, 0\right)$ and $\left(\sqrt{\frac{2}{3}}, 1\right)$. IP at $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{2}}\right)$

H.



28. $y = f(x) = x/\sqrt{x^2-1}$ **A.** $D = (-\infty,-1) \cup (1,\infty)$ **B.** No intercepts **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2-1}} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{x}{\sqrt{x^2-1}} = -1$, so $y = \pm 1$ are HA. $\lim\limits_{x\to 1^+} f(x) = +\infty$ and $\lim\limits_{x\to -1^-} f(x) = -\infty$, so $x = \pm 1$ are VA.

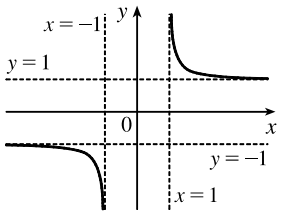
E. $f'(x) = \dfrac{\sqrt{x^2-1} - x\cdot\dfrac{x}{\sqrt{x^2-1}}}{[(x^2-1)^{1/2}]^2} = \dfrac{x^2-1-x^2}{(x^2-1)^{3/2}} = \dfrac{-1}{(x^2-1)^{3/2}} < 0$, so f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

F. No extreme values

G. $f''(x) = (-1)\left(-\frac{3}{2}\right)(x^2-1)^{-5/2}\cdot 2x = \dfrac{3x}{(x^2-1)^{5/2}}$.

$f''(x) < 0$ on $(-\infty,-1)$ and $f''(x) > 0$ on $(1,\infty)$, so f is CD on $(-\infty,-1)$ and CU on $(1,\infty)$. No IP

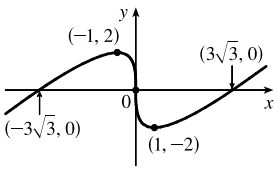
H.



29. $y = f(x) = x - 3x^{1/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = 3x^{1/3} \;\Rightarrow\; x^3 = 27x \;\Rightarrow\; x^3 - 27x = 0 \;\Rightarrow\; x(x^2-27) = 0 \;\Rightarrow\; x = 0, \pm 3\sqrt{3}$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin. **D.** No asymptote **E.** $f'(x) = 1 - x^{-2/3} = 1 - \dfrac{1}{x^{2/3}} = \dfrac{x^{2/3}-1}{x^{2/3}}$.

$f'(x) > 0$ when $|x| > 1$ and $f'(x) < 0$ when $0 < |x| < 1$, so f is increasing on $(-\infty,-1)$ and $(1,\infty)$, and decreasing on $(-1,0)$ and $(0,1)$ [hence decreasing on $(-1,1)$ since f is continuous on $(-1,1)$]. **F.** Local maximum value $f(-1) = 2$, local minimum value $f(1) = -2$ **G.** $f''(x) = \frac{2}{3}x^{-5/3} < 0$ when $x < 0$ and $f''(x) > 0$ when $x > 0$, so f is CD on $(-\infty,0)$ and CU on $(0,\infty)$. IP at $(0,0)$

H.



30. $y = f(x) = x^{5/3} - 5x^{2/3} = x^{2/3}(x-5)$ **A.** $D = \mathbb{R}$ **B.** x-intercepts 0, 5; y-intercept 0 **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty} x^{2/3}(x-5) = \pm\infty$, so there is no asymptote. **E.** $f'(x) = \frac{5}{3}x^{2/3} - \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x-2) > 0 \quad \Leftrightarrow$

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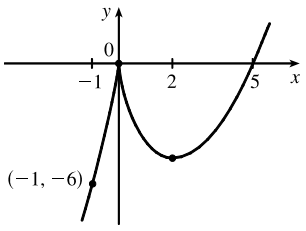
$x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$, $(2, \infty)$ and decreasing on $(0, 2)$.

F. Local maximum value $f(0) = 0$, local minimum value $f(2) = -3\sqrt[3]{4}$

G. $f''(x) = \frac{10}{9}x^{-1/3} + \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x+1) > 0 \quad \Leftrightarrow \quad x > -1$, so f is CU on $(-1, 0)$ and $(0, \infty)$, CD on $(-\infty, -1)$. IP at $(-1, -6)$

H.



31. $y = f(x) = \sqrt[3]{x^2 - 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -1$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x^2 - 1 = 0 \quad \Leftrightarrow \quad x = \pm 1$ **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

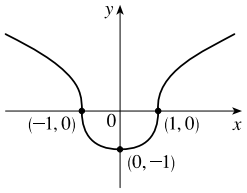
E. $f'(x) = \frac{1}{3}(x^2 - 1)^{-2/3}(2x) = \dfrac{2x}{3\sqrt[3]{(x^2-1)^2}}$. $f'(x) > 0 \quad \Leftrightarrow \quad x > 0$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$. **F.** Local minimum value $f(0) = -1$

G. $f''(x) = \dfrac{2}{3} \cdot \dfrac{(x^2-1)^{2/3}(1) - x \cdot \frac{2}{3}(x^2-1)^{-1/3}(2x)}{[(x^2-1)^{2/3}]^2}$

$= \dfrac{2}{9} \cdot \dfrac{(x^2-1)^{-1/3}[3(x^2-1) - 4x^2]}{(x^2-1)^{4/3}} = -\dfrac{2(x^2+3)}{9(x^2-1)^{5/3}}$

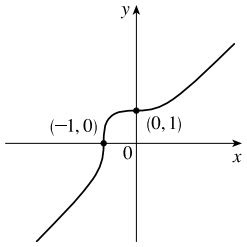
$f''(x) > 0 \quad \Leftrightarrow \quad -1 < x < 1$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(\pm 1, 0)$

H.



32. $y = f(x) = \sqrt[3]{x^3 + 1}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 1 = 0 \quad \Rightarrow \quad x = -1$ **C.** No symmetry **D.** No asymptote **E.** $f'(x) = \frac{1}{3}(x^3 + 1)^{-2/3}(3x^2) = \dfrac{x^2}{\sqrt[3]{(x^3+1)^2}}$. $f'(x) > 0$ if $x < -1$, $-1 < x < 0$, and $x > 0$, so f is increasing on $\mathbb{R}$. **F.** No local extrema

H.



G. $f''(x) = \dfrac{(x^3+1)^{2/3}(2x) - x^2 \cdot \frac{2}{3}(x^3+1)^{-1/3}(3x^2)}{[(x^3+1)^{2/3}]^2}$

$= \dfrac{x(x^3+1)^{-1/3}[2(x^3+1) - 2x^3]}{(x^3+1)^{4/3}} = \dfrac{2x}{(x^3+1)^{5/3}}$

$f''(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > 0$ and $f''(x) < 0 \quad \Leftrightarrow \quad -1 < x < 0$, so f is CU on $(-\infty, -1)$ and $(0, \infty)$ and CD on $(-1, 0)$. IP at $(-1, 0)$ and $(0, 1)$

33. $y = f(x) = \sin^3 x$ **A.** $D = \mathbb{R}$ **B.** x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = n\pi$, n an integer; y-intercept $= f(0) = 0$

C. $f(-x) = -f(x)$, so f is odd and the curve is symmetric about the origin. Also, $f(x + 2\pi) = f(x)$, so f is periodic with period 2π, and we determine **E–G** for $0 \le x \le \pi$. Since f is odd, we can reflect the graph of f on $[0, \pi]$ about the origin to obtain the graph of f on $[-\pi, \pi]$, and then since f has period 2π, we can extend the graph of f for all real numbers.

D. No asymptote **E.** $f'(x) = 3\sin^2 x \cos x > 0 \quad \Leftrightarrow \quad \cos x > 0$ and $\sin x \neq 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and f is decreasing on $\left(\frac{\pi}{2}, \pi\right)$. **F.** Local maximum value $f\left(\frac{\pi}{2}\right) = 1$ $\left[\text{local minimum value } f\left(-\frac{\pi}{2}\right) = -1\right]$

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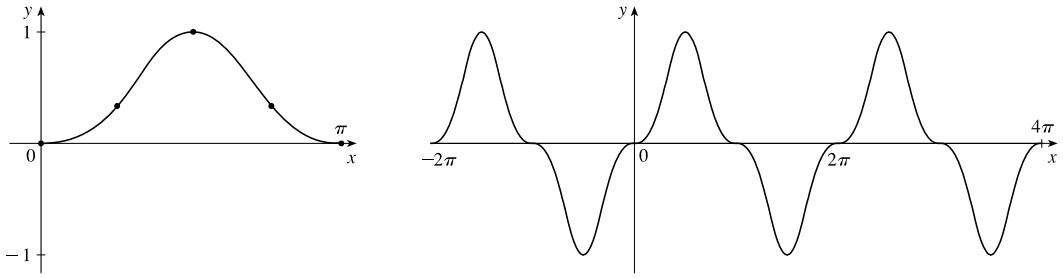
G. $f''(x) = 3\sin^2 x\,(-\sin x) + 3\cos x\,(2\sin x\,\cos x) = 3\sin x\,(2\cos^2 x - \sin^2 x)$

$= 3\sin x\,[2(1-\sin^2 x) - \sin^2 x] = 3\sin x(2 - 3\sin^2 x) > 0 \quad \Leftrightarrow$

$\sin x > 0$ and $\sin^2 x < \frac{2}{3} \quad \Leftrightarrow \quad 0 < x < \pi$ and $0 < \sin x < \sqrt{\frac{2}{3}} \quad \Leftrightarrow \quad 0 < x < \sin^{-1}\sqrt{\frac{2}{3}}$ $\left[\text{let } \alpha = \sin^{-1}\sqrt{\frac{2}{3}}\right]$ or $\pi - \alpha < x < \pi$, so f is CU on $(0, \alpha)$ and $(\pi - \alpha, \pi)$, and f is CD on $(\alpha, \pi - \alpha)$. There are inflection points at $x = 0$, π, α, and $x = \pi - \alpha$.

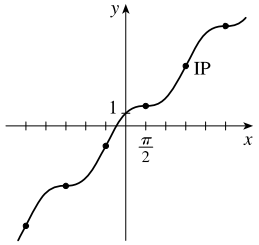
H.



34. $y = f(x) = x + \cos x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 1$; the x-intercept is about -0.74 and can be found using Newton's method **C.** No symmetry **D.** No asymptote **E.** $f'(x) = 1 - \sin x > 0$ except for $x = \frac{\pi}{2} + 2n\pi$, so f is increasing on $\mathbb{R}$. **F.** No local extrema

G. $f''(x) = -\cos x$. $f''(x) > 0 \quad \Rightarrow \quad -\cos x > 0 \quad \Rightarrow \quad \cos x < 0 \quad \Rightarrow$ x is in $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and $f''(x) < 0 \quad \Rightarrow$ x is in $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$, so f is CU on $\left(\frac{\pi}{2} + 2n\pi, \frac{3\pi}{2} + 2n\pi\right)$ and CD on $\left(-\frac{\pi}{2} + 2n\pi, \frac{\pi}{2} + 2n\pi\right)$. IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right) = \left(\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi\right)$ [on the line $y = x$]

H.

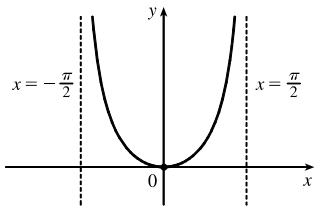


35. $y = f(x) = x\tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** Intercepts are 0 **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim_{x\to(\pi/2)^-} x\tan x = \infty$ and $\lim_{x\to-(\pi/2)^+} x\tan x = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA.

E. $f'(x) = \tan x + x\sec^2 x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f increases on $\left(0, \frac{\pi}{2}\right)$ and decreases on $\left(-\frac{\pi}{2}, 0\right)$. **F.** Absolute and local minimum value $f(0) = 0$.

G. $y'' = 2\sec^2 x + 2x\tan x\,\sec^2 x > 0$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. No IP

H.



36. $y = f(x) = 2x - \tan x, -\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow$ $2x = \tan x \quad \Leftrightarrow \quad x = 0$ or $x \approx \pm 1.17$ **C.** $f(-x) = -f(x)$, so f is odd; the graph is symmetric about the origin.

D. $\lim_{x\to(-\pi/2)^+} (2x - \tan x) = \infty$ and $\lim_{x\to(\pi/2)^-} (2x - \tan x) = -\infty$, so $x = \pm\frac{\pi}{2}$ are VA. No HA.

E. $f'(x) = 2 - \sec^2 x < 0 \quad \Leftrightarrow \quad |\sec x| > \sqrt{2}$ and $f'(x) > 0 \quad \Leftrightarrow \quad |\sec x| < \sqrt{2}$, so f is decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$,

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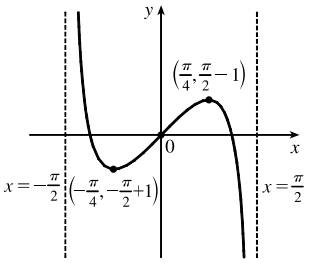
increasing on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, and decreasing again on $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ **F.** Local maximum value $f\left(\frac{\pi}{4}\right) = \frac{\pi}{2} - 1$, local minimum value $f\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2} + 1$

G. $f''(x) = -2\sec x \cdot \sec x \tan x = -2\tan x \sec^2 x = -2\tan x(\tan^2 x + 1)$ so $f''(x) > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, and $f''(x) < 0 \Leftrightarrow \tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$. Thus, f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

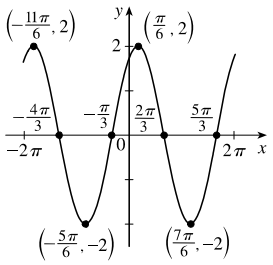
H.



37. $y = f(x) = \sin x + \sqrt{3}\cos x$, $-2\pi \le x \le 2\pi$ **A.** $D = [-2\pi, 2\pi]$ **B.** y-intercept: $f(0) = \sqrt{3}$; x-intercepts: $f(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow \tan x = -\sqrt{3} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$ **C.** f is periodic with period 2π. **D.** No asymptote **E.** $f'(x) = \cos x - \sqrt{3}\sin x$. $f'(x) = 0 \Leftrightarrow \cos x = \sqrt{3}\sin x \Leftrightarrow \tan x = \dfrac{1}{\sqrt{3}} \Leftrightarrow x = -\frac{11\pi}{6}, -\frac{5\pi}{6}, \frac{\pi}{6}$, or $\frac{7\pi}{6}$. $f'(x) < 0 \Leftrightarrow -\frac{11\pi}{6} < x < -\frac{5\pi}{6}$ or $\frac{\pi}{6} < x < \frac{7\pi}{6}$, so f is decreasing on $\left(-\frac{11\pi}{6}, -\frac{5\pi}{6}\right)$ and $\left(\frac{\pi}{6}, \frac{7\pi}{6}\right)$, and f is increasing on $\left(-2\pi, -\frac{11\pi}{6}\right)$, $\left(-\frac{5\pi}{6}, \frac{\pi}{6}\right)$, and $\left(\frac{7\pi}{6}, 2\pi\right)$. **F.** Local maximum value $f\left(-\frac{11\pi}{6}\right) = f\left(\frac{\pi}{6}\right) = \frac{1}{2} + \sqrt{3}\left(\frac{1}{2}\sqrt{3}\right) = 2$, local minimum value $f\left(-\frac{5\pi}{6}\right) = f\left(\frac{7\pi}{6}\right) = -\frac{1}{2} + \sqrt{3}\left(-\frac{1}{2}\sqrt{3}\right) = -2$

G. $f''(x) = -\sin x - \sqrt{3}\cos x$. $f''(x) = 0 \Leftrightarrow \sin x = -\sqrt{3}\cos x \Leftrightarrow \tan x = -\dfrac{1}{\sqrt{3}} \Leftrightarrow x = -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}$, or $\frac{5\pi}{3}$. $f''(x) > 0 \Leftrightarrow -\frac{4\pi}{3} < x < -\frac{\pi}{3}$ or $\frac{2\pi}{3} < x < \frac{5\pi}{3}$, so f is CU on $\left(-\frac{4\pi}{3}, -\frac{\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \frac{5\pi}{3}\right)$, and f is CD on $\left(-2\pi, -\frac{4\pi}{3}\right)$, $\left(-\frac{\pi}{3}, \frac{2\pi}{3}\right)$, and $\left(\frac{5\pi}{3}, 2\pi\right)$. There are IPs at $\left(-\frac{4\pi}{3}, 0\right)$, $\left(-\frac{\pi}{3}, 0\right)$, $\left(\frac{2\pi}{3}, 0\right)$, and $\left(\frac{5\pi}{3}, 0\right)$.

H.



38. $y = f(x) = \csc x - 2\sin x$, $0 < x < \pi$ **A.** $D = (0, \pi)$ **B.** No y-intercept; x-intercept: $f(x) = 0 \Leftrightarrow \csc x = 2\sin x \Leftrightarrow \frac{1}{2} = \sin^2 x \Leftrightarrow \sin x = \pm\frac{1}{2}\sqrt{2} \Leftrightarrow x = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ **C.** No symmetry

D. $\lim_{x \to 0^+} f(x) = \infty$ and $\lim_{x \to \pi^-} f(x) = \infty$, so $x = 0$ and $x = \pi$ are VAs.

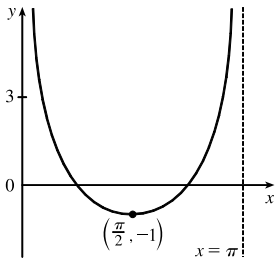
E. $f'(x) = -\csc x \cot x - 2\cos x = -\dfrac{\cos x}{\sin^2 x} - 2\cos x = -\cos x\left(\dfrac{1}{\sin^2 x} + 2\right)$. $f'(x) > 0$ when $-\cos x > 0 \Leftrightarrow \cos x < 0 \Leftrightarrow \frac{\pi}{2} < x < \pi$, so f' is increasing on $\left(\frac{\pi}{2}, \pi\right)$, and f is decreasing on $\left(0, \frac{\pi}{2}\right)$. **F.** Local minimum value $f\left(\frac{\pi}{2}\right) = -1$

G. $f''(x) = (-\csc x)(-\csc^2 x) + (\cot x)(\csc x \cot x) + 2\sin x$

$$= \frac{1 + \cos^2 x + 2\sin^4 x}{\sin^3 x}$$

f'' has the same sign as $\sin x$, which is positive on $(0, \pi)$, so f is CU on $(0, \pi)$. No IP

H.



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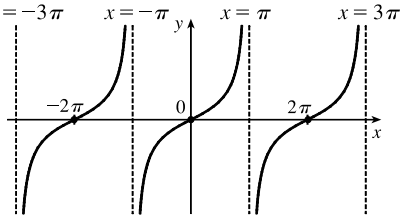
39. $y = f(x) = \dfrac{\sin x}{1+\cos x}$ $\left[\overset{\text{when } \cos x \neq 1}{=} \dfrac{\sin x}{1+\cos x} \cdot \dfrac{1-\cos x}{1-\cos x} = \dfrac{\sin x\,(1-\cos x)}{\sin^2 x} = \dfrac{1-\cos x}{\sin x} = \csc x - \cot x \right]$

A. The domain of f is the set of all real numbers except odd integer multiples of π; that is, all reals except $(2n+1)\pi$, where n is an integer. **B.** y-intercept: $f(0) = 0$; x-intercepts: $x = 2n\pi$, n an integer. **C.** $f(-x) = -f(x)$, so f is an odd function; the graph is symmetric about the origin and has period 2π. **D.** When n is an odd integer, $\lim\limits_{x \to (n\pi)^-} f(x) = \infty$ and $\lim\limits_{x \to (n\pi)^+} f(x) = -\infty$, so $x = n\pi$ is a VA for each odd integer n. No HA.

E. $f'(x) = \dfrac{(1+\cos x)\cdot\cos x - \sin x(-\sin x)}{(1+\cos x)^2} = \dfrac{1+\cos x}{(1+\cos x)^2} = \dfrac{1}{1+\cos x}$. $f'(x) > 0$ for all x except odd multiples of π, so f is increasing on $((2k-1)\pi, (2k+1)\pi)$ for each integer k. **F.** No extreme values

G. $f''(x) = \dfrac{\sin x}{(1+\cos x)^2} > 0 \quad \Rightarrow \quad \sin x > 0 \quad \Rightarrow$ $x \in (2k\pi, (2k+1)\pi)$ and $f''(x) < 0$ on $((2k-1)\pi, 2k\pi)$ for each integer k. f is CU on $(2k\pi, (2k+1)\pi)$ and CD on $((2k-1)\pi, 2k\pi)$ for each integer k. f has IPs at $(2k\pi, 0)$ for each integer k.

H.



40. $y = f(x) = \dfrac{\sin x}{2+\cos x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad \Leftrightarrow \quad x = n\pi$

C. $f(-x) = -f(x)$, so the curve is symmetric about the origin. f is periodic with period 2π, so we determine **E–G** for $0 \le x \le 2\pi$. **D.** No asymptote

E. $f'(x) = \dfrac{(2+\cos x)\cos x - \sin x(-\sin x)}{(2+\cos x)^2} = \dfrac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} = \dfrac{2\cos x + 1}{(2+\cos x)^2}$.

$f'(x) > 0 \quad \Leftrightarrow \quad 2\cos x + 1 > 0 \quad \Leftrightarrow \quad \cos x > -\frac{1}{2} \quad \Leftrightarrow \quad x$ is in $\left(0, \frac{2\pi}{3}\right)$ or $\left(\frac{4\pi}{3}, 2\pi\right)$, so f is increasing on $\left(0, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, 2\pi\right)$, and f is decreasing on $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$.

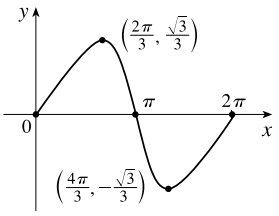
F. Local maximum value $f\left(\frac{2\pi}{3}\right) = \dfrac{\sqrt{3}/2}{2-(1/2)} = \dfrac{\sqrt{3}}{3}$ and local minimum value $f\left(\frac{4\pi}{3}\right) = \dfrac{-\sqrt{3}/2}{2-(1/2)} = -\dfrac{\sqrt{3}}{3}$

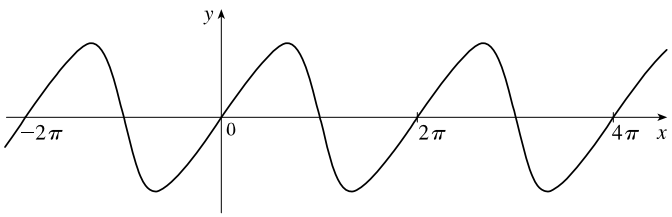
G. $f''(x) = \dfrac{(2+\cos x)^2(-2\sin x) - (2\cos x + 1)2(2+\cos x)(-\sin x)}{[(2+\cos x)^2]^2}$

$= \dfrac{-2\sin x\,(2+\cos x)[(2+\cos x) - (2\cos x + 1)]}{(2+\cos x)^4} = \dfrac{-2\sin x\,(1-\cos x)}{(2+\cos x)^3}$

$f''(x) > 0 \quad \Leftrightarrow \quad -2\sin x > 0 \quad \Leftrightarrow \quad \sin x < 0 \quad \Leftrightarrow \quad x$ is in $(\pi, 2\pi)$ [f is CU] and $f''(x) < 0 \quad \Leftrightarrow$ x is in $(0, \pi)$ [f is CD]. The inflection points are $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$.

H.





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41. $y = f(x) = \arctan(e^x)$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = \arctan 1 = \frac{\pi}{4}$. $f(x) > 0$ so there are no x-intercepts.

C. No symmetry **D.** $\lim_{x\to-\infty} \arctan(e^x) = 0$ and $\lim_{x\to\infty} \arctan(e^x) = \frac{\pi}{2}$, so $y = 0$ and $y = \frac{\pi}{2}$ are HAs. No VA

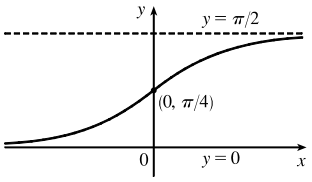
E. $f'(x) = \dfrac{1}{1+(e^x)^2}\dfrac{d}{dx}e^x = \dfrac{e^x}{1+e^{2x}} > 0$, so f is increasing on $(-\infty, \infty)$. **F.** No local extrema

G. $f''(x) = \dfrac{(1+e^{2x})e^x - e^x(2e^{2x})}{(1+e^{2x})^2} = \dfrac{e^x[(1+e^{2x}) - 2e^{2x}]}{(1+e^{2x})^2}$

$= \dfrac{e^x(1-e^{2x})}{(1+e^{2x})^2} > 0 \quad \Leftrightarrow$

$1 - e^{2x} > 0 \quad \Leftrightarrow \quad e^{2x} < 1 \quad \Leftrightarrow \quad 2x < 0 \quad \Leftrightarrow \quad x < 0$, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{\pi}{4}\right)$

H.



42. $y = f(x) = (1-x)e^x$ **A.** $D = \mathbb{R}$ **B.** x-intercept 1, y-intercept $= f(0) = 1$ **C.** No symmetry

D. $\lim_{x\to-\infty} \dfrac{1-x}{e^{-x}}$ $\left[\text{form } \frac{\infty}{\infty}\right] \overset{\text{H}}{=} \lim_{x\to-\infty} \dfrac{-1}{-e^{-x}} = 0$, so $y = 0$ is a HA. No VA

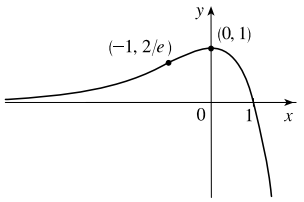
E. $f'(x) = (1-x)e^x + e^x(-1) = e^x[(1-x)+(-1)] = -xe^x > 0 \quad \Leftrightarrow \quad x < 0$, so f is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

F. Local maximum value $f(0) = 1$, no local minimum value

G. $f''(x) = -xe^x + e^x(-1) = e^x(-x-1) = -(x+1)e^x > 0 \quad \Leftrightarrow$

$x < -1$, so f is CU on $(-\infty, -1)$ and CD on $(-1, \infty)$. IP at $(-1, 2/e)$

H.



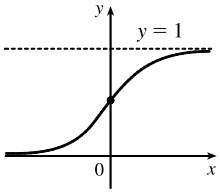
43. $y = 1/(1+e^{-x})$ **A.** $D = \mathbb{R}$ **B.** No x-intercept; y-intercept $= f(0) = \frac{1}{2}$. **C.** No symmetry

D. $\lim_{x\to\infty} 1/(1+e^{-x}) = \frac{1}{1+0} = 1$ and $\lim_{x\to-\infty} 1/(1+e^{-x}) = 0$ since $\lim_{x\to-\infty} e^{-x} = \infty$, so f has horizontal asymptotes $y = 0$ and $y = 1$. **E.** $f'(x) = -(1+e^{-x})^{-2}(-e^{-x}) = e^{-x}/(1+e^{-x})^2$. This is positive for all x, so f is increasing on $\mathbb{R}$.

F. No extreme values **G.** $f''(x) = \dfrac{(1+e^{-x})^2(-e^{-x}) - e^{-x}(2)(1+e^{-x})(-e^{-x})}{(1+e^{-x})^4} = \dfrac{e^{-x}(e^{-x}-1)}{(1+e^{-x})^3}$

The second factor in the numerator is negative for $x > 0$ and positive for $x < 0$, and the other factors are always positive, so f is CU on $(-\infty, 0)$ and CD on $(0, \infty)$. IP at $\left(0, \frac{1}{2}\right)$

H.



44. $y = f(x) = e^{-x}\sin x,\ 0 \le x \le 2\pi$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad \sin x = 0 \quad \Leftrightarrow$ $x = 0, \pi$, and 2π. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^{-x}\cos x + \sin x\,(-e^{-x}) = e^{-x}(\cos x - \sin x)$. $f'(x) = 0 \quad \Leftrightarrow \quad \cos x = \sin x \quad \Leftrightarrow \quad x = \frac{\pi}{4}, \frac{5\pi}{4}$. $f'(x) > 0$ if x is in $\left(0, \frac{\pi}{4}\right)$ or $\left(\frac{5\pi}{4}, 2\pi\right)$ [f is increasing] and $f'(x) < 0$ if x is in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ [f is decreasing]. **F.** Local maximum value $f\left(\frac{\pi}{4}\right)$ and local minimum value $f\left(\frac{5\pi}{4}\right)$

G. $f''(x) = e^{-x}(-\sin x - \cos x) + (\cos x - \sin x)(-e^{-x}) = e^{-x}(-2\cos x)$. $f''(x) > 0 \quad \Leftrightarrow \quad -2\cos x > 0 \quad \Leftrightarrow$

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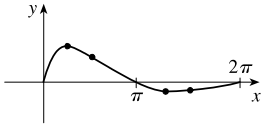
$\cos x < 0 \;\Rightarrow\; x$ is in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ [f is CU] and $f''(x) < 0 \;\Leftrightarrow$

$\cos x > 0 \;\Rightarrow\; x$ is in $\left(0, \frac{\pi}{2}\right)$ or $\left(\frac{3\pi}{2}, 2\pi\right)$ [f is CD].

IP at $\left(\frac{\pi}{2} + n\pi, f\left(\frac{\pi}{2} + n\pi\right)\right)$

H.



45. $y = f(x) = \dfrac{1}{x} + \ln x$ **A.** $D = (0, \infty)$ [same as $\ln x$] **B.** No y-intercept; no x-intercept [$1/x > |\ln x|$ on $(0, 1)$, and $1/x$ and $\ln x$ are both positive on $(1, \infty)$] **C.** No symmetry **D.** $\lim\limits_{x \to 0^+} f(x) = \infty$, so $x = 0$ is a VA.

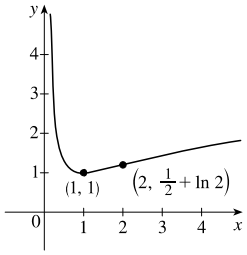
E. $f'(x) = -\dfrac{1}{x^2} + \dfrac{1}{x} = \dfrac{x - 1}{x^2}$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and f is decreasing on $(0, 1)$.

F. Local minimum value $f(1) = 1$ **G.** $f''(x) = \dfrac{2}{x^3} - \dfrac{1}{x^2} = \dfrac{2 - x}{x^3}$.

$f''(x) > 0$ for $0 < x < 2$, so f is CU on $(0, 2)$, and f is CD on $(2, \infty)$.

IP at $\left(2, \frac{1}{2} + \ln 2\right)$

H.



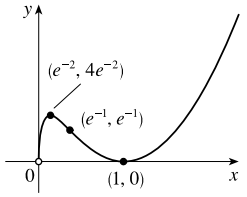
46. $y = x\,(\ln x)^2$ **A.** $D = (0, \infty)$ [same as $\ln x$] **B.** No y-intercept; x-intercepts: $f(x) = 0 \;\Rightarrow$ $(\ln x)^2 = 0$ [$x = 0$ not in domain] $\;\Rightarrow\; \ln x = 0 \;\Rightarrow\; x = 1$ **C.** No symmetry **D.** No HA;

$\lim\limits_{x \to 0^+} x\,(\ln x)^2 = \lim\limits_{x \to 0^+} \dfrac{(\ln x)^2}{1/x} \overset{\text{H}}{=} \lim\limits_{x \to 0^+} \dfrac{2 \ln x \cdot 1/x}{-1/x^2} = \lim\limits_{x \to 0^+} \dfrac{2 \ln x}{-1/x} \overset{\text{H}}{=} \lim\limits_{x \to 0^+} \dfrac{2/x}{1/x^2} = \lim\limits_{x \to 0^+} (2x) = 0$, no VA.

E. $f'(x) = x \cdot 2 \ln x \cdot \left(\dfrac{1}{x}\right) + (\ln x)^2 \cdot 1 = 2 \ln x + (\ln x)^2 = \ln x\,(2 + \ln x) = 0 \;\Leftrightarrow\; x = 1$ or $x = e^{-2}$;

$f'(x) > 0 \;\Leftrightarrow\; 0 < x < e^{-2}$ or $x > 1$ and $f'(x) < 0 \;\Leftrightarrow\; e^{-2} < x < 1$, so f is increasing on $(0, e^{-2})$ and $(1, \infty)$, and decreasing on $\left(e^{-2}, 1\right)$. **F.** Local maximum value $f(e^{-2}) = 4e^{-2}$, local minimum value $f(1) = 0$ **G.** $f''(x) = \dfrac{2}{x} + 2\,(\ln x) \cdot \dfrac{1}{x} = \dfrac{2\,(1 + \ln x)}{x}$, so $f''(x) > 0 \;\Leftrightarrow\; 1 + \ln x > 0 \;\Leftrightarrow\; x > e^{-1}$ and $f''(x) < 0 \;\Leftrightarrow$ $0 < x < e^{-1}$. Thus, f is CD on $(0, e^{-1})$ and CU on (e^{-1}, ∞). IP at (e^{-1}, e^{-1})

H.



47. $y = f(x) = (1 + e^x)^{-2} = \dfrac{1}{(1 + e^x)^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = \frac{1}{4}$. x-intercepts: none [since $f(x) > 0$]

C. No symmetry **D.** $\lim\limits_{x \to \infty} f(x) = 0$ and $\lim\limits_{x \to -\infty} f(x) = 1$, so $y = 0$ and $y = 1$ are HA; no VA

E. $f'(x) = -2(1 + e^x)^{-3} e^x = \dfrac{-2e^x}{(1 + e^x)^3} < 0$, so f is decreasing on $\mathbb{R}$ **F.** No local extrema

G. $f''(x) = (1 + e^x)^{-3}(-2e^x) + (-2e^x)(-3)(1 + e^x)^{-4} e^x$

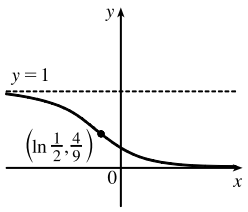
$= -2e^x (1 + e^x)^{-4}[(1 + e^x) - 3e^x] = \dfrac{-2e^x (1 - 2e^x)}{(1 + e^x)^4}$.

$f''(x) > 0 \;\Leftrightarrow\; 1 - 2e^x < 0 \;\Leftrightarrow\; e^x > \frac{1}{2} \;\Leftrightarrow\; x > \ln \frac{1}{2}$ and

$f''(x) < 0 \;\Leftrightarrow\; x < \ln \frac{1}{2}$, so f is CU on $\left(\ln \frac{1}{2}, \infty\right)$ and CD on $\left(-\infty, \ln \frac{1}{2}\right)$.

IP at $\left(\ln \frac{1}{2}, \frac{4}{9}\right)$

H.



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48. $y = f(x) = e^x/x^2$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept **C.** No symmetry **D.** $\lim_{x\to-\infty} \dfrac{e^x}{x^2} = 0$, so $y = 0$ is HA. $\lim_{x\to 0} \dfrac{e^x}{x^2} = \infty$, so $x = 0$ is a VA. **E.** $f'(x) = \dfrac{x^2e^x - e^x(2x)}{(x^2)^2} = \dfrac{xe^x(x-2)}{x^4} = \dfrac{e^x(x-2)}{x^3} > 0 \quad \Leftrightarrow \quad x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$.

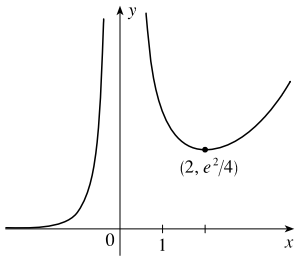
F. Local minimum value $f(2) = e^2/4 \approx 1.85$, no local maximum value

G. $f''(x) = \dfrac{x^3[e^x(1) + (x-2)e^x] - e^x(x-2)(3x^2)}{(x^3)^2}$

$= \dfrac{x^2e^x[x(x-1) - 3(x-2)]}{x^6} = \dfrac{e^x(x^2 - 4x + 6)}{x^4} > 0$

for all x in the domain of f; that is, f is CU on $(-\infty, 0)$ and $(0, \infty)$. No IP

H.



49. $y = f(x) = \ln(\sin x)$

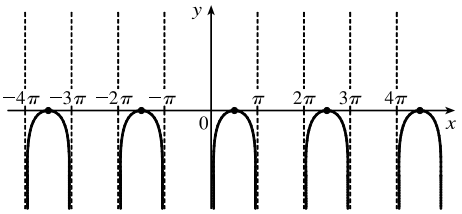
A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\,\pi) = \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad \ln(\sin x) = 0 \quad \Leftrightarrow \quad \sin x = e^0 = 1 \quad \Leftrightarrow \quad x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n.

F. Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum.

G. $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.



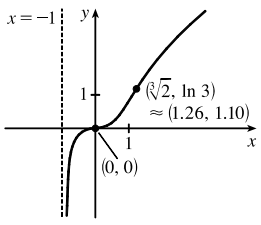
50. $y = f(x) = \ln(1 + x^3)$ **A.** $1 + x^3 > 0 \quad \Leftrightarrow \quad x^3 > -1 \quad \Leftrightarrow \quad x > -1$, so $D = (-1, \infty)$. **B.** y-intercept: $f(0) = \ln 1 = 0$; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad \ln(1 + x^3) = 0 \quad \Leftrightarrow \quad 1 + x^3 = e^0 \quad \Leftrightarrow \quad x^3 = 0 \quad \Leftrightarrow \quad x = 0$ **C.** No symmetry **D.** $\lim_{x\to-1^+} f(x) = -\infty$, so $x = -1$ is a VA **E.** $f'(x) = \dfrac{3x^2}{1 + x^3}$. $f'(x) > 0$ on $(-1, 0)$ and $(0, \infty)$ [$f'(x) = 0$ at $x = 0$], so by Exercise 4.3.97, f is increasing on $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{(1 + x^3)(6x) - 3x^2(3x^2)}{(1 + x^3)^2}$

$= \dfrac{3x[2(1 + x^3) - 3x^3]}{(1 + x^3)^2} = \dfrac{3x(2 - x^3)}{(1 + x^3)^2}$

$f''(x) > 0 \quad \Leftrightarrow \quad 0 < x < \sqrt[3]{2}$, so f is CU on $\left(0, \sqrt[3]{2}\right)$ and f is CD on $(-1, 0)$ and $\left(\sqrt[3]{2}, \infty\right)$. IP at $(0, 0)$ and $\left(\sqrt[3]{2}, \ln 3\right)$

H.



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51. $y = f(x) = xe^{-1/x}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** No intercept **C.** No symmetry

D. $\lim_{x\to 0^-} \dfrac{e^{-1/x}}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^-} \dfrac{e^{-1/x}(1/x^2)}{-1/x^2} = -\lim_{x\to 0^-} e^{-1/x} = -\infty$, so $x = 0$ is a VA. Also, $\lim_{x\to 0^+} xe^{-1/x} = 0$, so the graph approaches the origin as $x \to 0^+$. **E.** $f'(x) = xe^{-1/x}\left(\dfrac{1}{x^2}\right) + e^{-1/x}(1) = e^{-1/x}\left(\dfrac{1}{x} + 1\right) = \dfrac{x+1}{xe^{1/x}} > 0 \quad \Leftrightarrow$ $x < -1$ or $x > 0$, so f is increasing on $(-\infty, -1)$ and $(0, \infty)$, and f is decreasing on $(-1, 0)$.

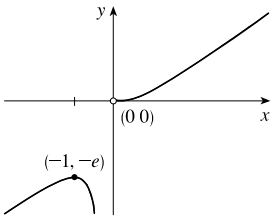
F. Local maximum value $f(-1) = -e$, no local minimum value

G. $f'(x) = e^{-1/x}\left(\dfrac{1}{x} + 1\right) \quad \Rightarrow$

$$f''(x) = e^{-1/x}\left(-\frac{1}{x^2}\right) + \left(\frac{1}{x} + 1\right)e^{-1/x}\left(\frac{1}{x^2}\right)$$
$$= \frac{1}{x^2}e^{-1/x}\left[-1 + \left(\frac{1}{x} + 1\right)\right] = \frac{1}{x^3 e^{1/x}} > 0 \quad \Leftrightarrow$$

$x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

H.



52. $y = f(x) = \dfrac{\ln x}{x^2}$ **A.** $D = (0, \infty)$ **B.** y-intercept: none; x-intercept: $f(x) = 0 \quad \Leftrightarrow \quad \ln x = 0 \quad \Leftrightarrow \quad x = 1$

C. No symmetry **D.** $\lim_{x\to 0^+} f(x) = -\infty$, so $x = 0$ is a VA; $\lim_{x\to\infty} \dfrac{\ln x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} \dfrac{1/x}{2x} = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1 - 2\ln x)}{x^4} = \dfrac{1 - 2\ln x}{x^3}$. $f'(x) > 0 \quad \Leftrightarrow \quad 1 - 2\ln x > 0 \quad \Leftrightarrow \quad \ln x < \frac{1}{2} \quad \Rightarrow$ $0 < x < e^{1/2}$ and $f'(x) < 0 \quad \Rightarrow \quad x > e^{1/2}$, so f is increasing on $\left(0, \sqrt{e}\right)$ and decreasing on $\left(\sqrt{e}, \infty\right)$.

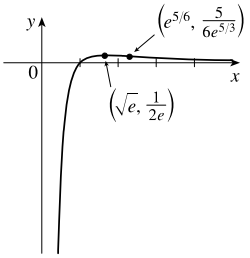
F. Local maximum value $f(e^{1/2}) = \dfrac{1/2}{e} = \dfrac{1}{2e}$

G. $f''(x) = \dfrac{x^3(-2/x) - (1 - 2\ln x)(3x^2)}{(x^3)^2}$

$$= \frac{x^2[-2 - 3(1 - 2\ln x)]}{x^6} = \frac{-5 + 6\ln x}{x^4}$$

$f''(x) > 0 \quad \Leftrightarrow \quad -5 + 6\ln x > 0 \quad \Leftrightarrow \quad \ln x > \frac{5}{6} \quad \Rightarrow \quad x > e^{5/6}$ [f is CU] and $f''(x) < 0 \quad \Leftrightarrow \quad 0 < x < e^{5/6}$ [f is CD]. IP at $(e^{5/6}, 5/(6e^{5/3}))$

H.



53. $y = f(x) = e^{\arctan x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = e^0 = 1$; no x-intercept since $e^{\arctan x}$ is positive for all x.

C. No symmetry **D.** $\lim_{x\to-\infty} f(x) = e^{-\pi/2}$ [≈ 0.21], so $y = e^{-\pi/2}$ is a HA. $\lim_{x\to\infty} f(x) = e^{\pi/2}$ [≈ 4.81], so $y = e^{\pi/2}$ is a HA. **E.** $f'(x) = e^{\arctan x}\left(\dfrac{1}{1 + x^2}\right)$. $f'(x) > 0$ for all x, so f is increasing on $\mathbb{R}$. **F.** No local extrema

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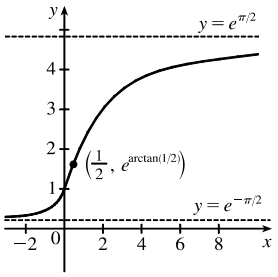
G. $f''(x) = \dfrac{(1+x^2)e^{\arctan x}\left(\dfrac{1}{1+x^2}\right) - e^{\arctan x}(2x)}{(1+x^2)^2}$

$= \dfrac{e^{\arctan x}(1-2x)}{(1+x^2)^2}$

$f''(x) > 0$ for $x < \frac{1}{2}$, so f is CU on $\left(-\infty, \frac{1}{2}\right)$ and f is CD on $\left(\frac{1}{2}, \infty\right)$.

IP at $\left(\frac{1}{2}, e^{\arctan 1/2}\right) \approx (0.5, 1.59)$

H.



54. $y = f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \neq -1\}$ **B.** x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$

C. No symmetry **D.** $\lim\limits_{x\to\pm\infty} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \lim\limits_{x\to\pm\infty} \tan^{-1}\left(\dfrac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA.

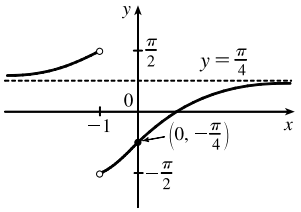
Also $\lim\limits_{x\to-1^+} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = -\dfrac{\pi}{2}$ and $\lim\limits_{x\to-1^-} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \dfrac{\pi}{2}$.

E. $f'(x) = \dfrac{1}{1+[(x-1)/(x+1)]^2}\dfrac{(x+1)-(x-1)}{(x+1)^2} = \dfrac{2}{(x+1)^2+(x-1)^2} = \dfrac{1}{x^2+1} > 0,$

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = -2x/\left(x^2+1\right)^2 > 0 \quad \Leftrightarrow \quad x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

H.



55. $g(x) = \sqrt{f(x)}$

(a) The domain of g consists of all x such that $f(x) \geq 0$, so g has domain $(-\infty, 7]$. $g'(x) = \dfrac{1}{2\sqrt{f(x)}} \cdot f'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

Since $f'(3)$ does not exist, $g'(3)$ does not exist. (Note that $f(7) = 0$, but 7 is an endpoint of the domain of g.) The domain of g' is $(-\infty, 3) \cup (3, 7)$.

(b) $g'(x) = 0 \quad \Rightarrow \quad f'(x) = 0$ on the domain of $g \quad \Rightarrow \quad x = 5$ [there is a horizontal tangent line there]. From part (a), $g'(3)$ does not exist. So the critical numbers of g are 3 and 5.

(c) From part (a), $g'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$. $g'(6) = \dfrac{f'(6)}{2\sqrt{f(6)}} \approx \dfrac{-2}{2\sqrt{3}} = -\dfrac{1}{\sqrt{3}} \approx -0.58$.

(d) $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \sqrt{f(x)} = \sqrt{2}$, so $y = \sqrt{2}$ is a horizontal asymptote. No VA

56. $g(x) = \sqrt[3]{f(x)}$

(a) Since the cube-root function is defined for all reals, the domain of g equals the domain of f, $(-\infty, \infty)$.

$g'(x) = \dfrac{1}{3\left(\sqrt[3]{f(x)}\right)^2} \cdot f'(x) = \dfrac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. Since $f'(3)$ does not exist and $f(7) = 0$, $g'(3)$ and $g'(7)$ do not exist.

The domain of g' is $(-\infty, 3) \cup (3, 7) \cup (7, \infty)$.

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(b) $g'(x)=0 \;\Rightarrow\; f'(x)=0 \;\Rightarrow\; x=5$ or $x=9$ [there are horizontal tangent lines there]. From part (a), $g'(x)$ does not exist at $x=3$ and $x=7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) From part (a), $g'(x)=\dfrac{f'(x)}{3\left(\sqrt[3]{f(x)}\right)^2}$. $g'(6)=\dfrac{f'(6)}{3\left(\sqrt[3]{f(6)}\right)^2}\approx\dfrac{-2}{3\left(\sqrt[3]{3}\right)^2}=-\dfrac{2}{3^{5/3}}\approx -0.32$.

(d) $\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}\sqrt[3]{f(x)}=\sqrt[3]{2}$ and $\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}\sqrt[3]{f(x)}=\sqrt[3]{-1}=-1$, so $y=\sqrt[3]{2}$ and $y=-1$ are horizontal asymptotes. No VA

57. $g(x)=|f(x)|$

(a) Since the absolute-value function is defined for all reals, the domain of g equals the domain of f, $(-\infty,\infty)$. The domain of g' equals the domain of f' except for any values of x such that both $f(x)=0$ and $f'(x)\neq 0$. $f'(3)$ does not exist, $f(7)=0$, and $f'(7)\neq 0$. Thus, the domain of g' is $(-\infty,3)\cup(3,7)\cup(7,\infty)$.

(b) $g'(x)=0 \;\Rightarrow\; f'(x)=0 \;\Rightarrow\; x=5$ or $x=9$ [there are horizontal tangent lines there]. From part (a), g' does not exist at $x=3$ and $x=7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) Since f is positive near $x=6$, $g(x)=|f(x)|=f(x)$ near 6, so $g'(6)=f'(6)\approx -2$.

(d) $\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}|f(x)|=|2|=2$ and $\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}|f(x)|=|-1|=1$, so $y=2$ and $y=1$ are horizontal asymptotes. No VA

58. $g(x)=1/f(x)$

(a) The domain of g consists of all x such that $f(x)\neq 0$, so g has domain $(-\infty,7)\cup(7,\infty)$. $g(x)=\dfrac{1}{f(x)}=[f(x)]^{-1} \;\Rightarrow$ $g'(x)=-1[f(x)]^{-2}\cdot f'(x)=-\dfrac{f'(x)}{[f(x)]^2}$. The domain of g' will equal the domain of f except for any values of f such that $f(x)=0$. $f'(3)$ does not exist, and $f(7)=0$. Thus, the domain of g' is $(-\infty,3)\cup(3,7)\cup(7,\infty)$.

(b) $g'(x)=0 \;\Rightarrow\; f'(x)=0 \;\Rightarrow\; x=5$ or $x=9$ [there are horizontal tangent lines there]. From part (a), g' does not exist at $x=3$ and $x=7$. So the critical numbers of g are 3, 5, 7, and 9.

(c) From part (a), $g'(x)=-\dfrac{f'(x)}{[f(x)]^2}$. $g'(6)=-\dfrac{f'(6)}{[f(6)]^2}\approx -\left(\dfrac{-2}{3^2}\right)=\dfrac{2}{9}$.

(d) $\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}\dfrac{1}{f(x)}=\dfrac{1}{2}$ and $\lim\limits_{x\to\infty} g(x)=\lim\limits_{x\to\infty}\dfrac{1}{f(x)}=\dfrac{1}{-1}=-1$, so $y=\dfrac{1}{2}$ and $y=-1$ are horizontal asymptotes. $\lim\limits_{x\to7^-}\dfrac{1}{f(x)}=\dfrac{1}{\lim\limits_{x\to7^-}f(x)}=\infty$ and $\lim\limits_{x\to7^+} g(x)=\lim\limits_{x\to7^+}\dfrac{1}{f(x)}=\dfrac{1}{\lim\limits_{x\to7^+}f(x)}=-\infty$, so $x=7$ is a vertical asymptote.

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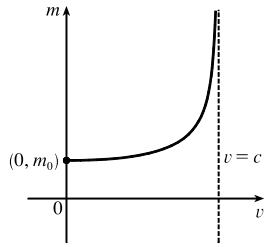
59. $m = f(v) = \dfrac{m_0}{\sqrt{1 - v^2/c^2}}$. The m-intercept is $f(0) = m_0$. There are no v-intercepts. $\lim\limits_{v \to c^-} f(v) = \infty$, so $v = c$ is a VA.

$f'(v) = -\frac{1}{2} m_0 (1 - v^2/c^2)^{-3/2}(-2v/c^2) = \dfrac{m_0 v}{c^2(1 - v^2/c^2)^{3/2}} = \dfrac{m_0 v}{\dfrac{c^2(c^2 - v^2)^{3/2}}{c^3}} = \dfrac{m_0 c v}{(c^2 - v^2)^{3/2}} > 0$, so f is increasing on $(0, c)$. There are no local extreme values.



$$f''(v) = \frac{(c^2 - v^2)^{3/2}(m_0 c) - m_0 c v \cdot \frac{3}{2}(c^2 - v^2)^{1/2}(-2v)}{[(c^2 - v^2)^{3/2}]^2}$$

$$= \frac{m_0 c (c^2 - v^2)^{1/2}[(c^2 - v^2) + 3v^2]}{(c^2 - v^2)^3} = \frac{m_0 c (c^2 + 2v^2)}{(c^2 - v^2)^{5/2}} > 0,$$

so f is CU on $(0, c)$. There are no inflection points.

60. Let $a = m_0^2 c^4$ and $b = h^2 c^2$, so the equation can be written as $E = f(\lambda) = \sqrt{a + b/\lambda^2} = \sqrt{\dfrac{a\lambda^2 + b}{\lambda^2}} = \dfrac{\sqrt{a\lambda^2 + b}}{\lambda}$.

$\lim\limits_{\lambda \to 0^+} \dfrac{\sqrt{a\lambda^2 + b}}{\lambda} = \infty$, so $\lambda = 0$ is a VA.

$\lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a\lambda^2 + b}}{\lambda} = \lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a\lambda^2 + b}/\lambda}{\lambda/\lambda} = \lim\limits_{\lambda \to \infty} \dfrac{\sqrt{a + b/\lambda^2}}{1} = \sqrt{a}$, so $E = \sqrt{a} = m_0 c^2$ is a HA.

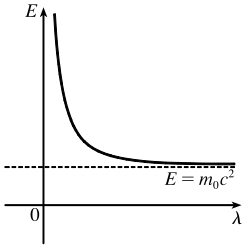
$$f'(\lambda) = \frac{\lambda \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) - (a\lambda^2 + b)^{1/2}(1)}{\lambda^2} = \frac{(a\lambda^2 + b)^{-1/2}[a\lambda^2 - (a\lambda^2 + b)]}{\lambda^2} = \frac{-b}{\lambda^2 \sqrt{a\lambda^2 + b}} < 0,$$

so f is decreasing on $(0, \infty)$. Using the Reciprocal Rule,

$$f''(\lambda) = b \cdot \frac{\lambda^2 \cdot \frac{1}{2}(a\lambda^2 + b)^{-1/2}(2a\lambda) + (a\lambda^2 + b)^{1/2}(2\lambda)}{\left(\lambda^2 \sqrt{a\lambda^2 + b}\right)^2}$$

$$= \frac{b\lambda(a\lambda^2 + b)^{-1/2}[a\lambda^2 + 2(a\lambda^2 + b)]}{\left(\lambda^2 \sqrt{a\lambda^2 + b}\right)^2} = \frac{b(3a\lambda^2 + 2b)}{\lambda^3 (a\lambda^2 + b)^{3/2}} > 0,$$

so f is CU on $(0, \infty)$. There are no extrema or inflection points. The graph shows that as λ decreases, the energy increases and as λ increases, the energy decreases. For large wavelengths, the energy is very close to the energy at rest.



61. (a) $p(t) = \dfrac{1}{2} \;\Rightarrow\; \dfrac{1}{2} = \dfrac{1}{1 + ae^{-kt}} \;\Leftrightarrow\; 1 + ae^{-kt} = 2 \;\Leftrightarrow\; ae^{-kt} = 1 \;\Leftrightarrow\; e^{-kt} = \dfrac{1}{a} \;\Leftrightarrow$

$\ln e^{-kt} = \ln a^{-1} \;\Leftrightarrow\; -kt = -\ln a \;\Leftrightarrow\; t = \dfrac{\ln a}{k}$, which is when half the population will have heard the rumor.

(b) The rate of spread is given by $p'(t) = \dfrac{ake^{-kt}}{(1 + ae^{-kt})^2}$. To find the greatest rate of spread, we'll apply the First Derivative Test to $p'(t)$ [not $p(t)$].

$$[p'(t)]' = p''(t) = \frac{(1 + ae^{-kt})^2(-ak^2 e^{-kt}) - ake^{-kt} \cdot 2(1 + ae^{-kt})(-ake^{-kt})}{[(1 + ae^{-kt})^2]^2}$$

$$= \frac{(1 + ae^{-kt})(-ake^{-kt})[k(1 + ae^{-kt}) - 2ake^{-kt}]}{(1 + ae^{-kt})^4} = \frac{-ake^{-kt}(k)(1 - ae^{-kt})}{(1 + ae^{-kt})^3} = \frac{ak^2 e^{-kt}(ae^{-kt} - 1)}{(1 + ae^{-kt})^3}$$

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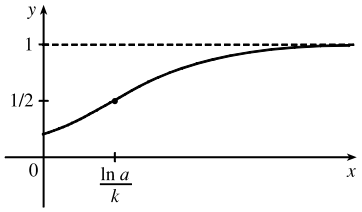
$p''(t) > 0 \quad \Leftrightarrow \quad ae^{-kt} > 1 \quad \Leftrightarrow \quad -kt > \ln a^{-1} \quad \Leftrightarrow \quad t < \dfrac{\ln a}{k}$, so $p'(t)$ is increasing for $t < \dfrac{\ln a}{k}$ and $p'(t)$ is decreasing for $t > \dfrac{\ln a}{k}$. Thus, $p'(t)$, the rate of spread of the rumor, is greatest at the same time, $\dfrac{\ln a}{k}$, as when half the population [by part (a)] has heard it.

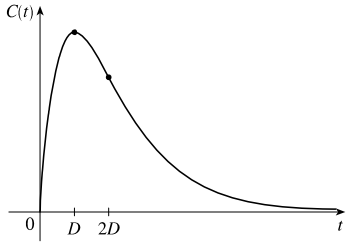
(c) $p(0) = \dfrac{1}{1+a}$ and $\lim\limits_{t\to\infty} p(t) = 1$. The graph is shown with $a = 4$ and $k = \frac{1}{2}$.



62. $C(t) = K(e^{-at} - e^{-bt})$, where $K > 0$ and $b > a > 0$. $C(0) = K(1-1) = 0$ is the only intercept. $\lim\limits_{t\to\infty} C(t) = 0$, so $C = 0$ is a HA. $C'(t) = K(-ae^{-at} + be^{-bt}) > 0 \quad \Leftrightarrow \quad be^{-bt} > ae^{-at} \quad \Leftrightarrow \quad e^{at}e^{-bt} > \dfrac{a}{b} \quad \Leftrightarrow \quad e^{(a-b)t} > \dfrac{a}{b} \quad \Leftrightarrow$

$(a-b)t > \ln\dfrac{a}{b} \quad \Leftrightarrow \quad t > \dfrac{\ln(a/b)}{a-b}$ or $\dfrac{\ln(b/a)}{b-a}$ [call this value D]. C is increasing for $t < D$ and decreasing for $t > D$, so $C(D)$ is a local maximum [and absolute] value. $C''(t) = K(a^2e^{-at} - b^2e^{-bt}) > 0 \quad \Leftrightarrow \quad a^2e^{-at} > b^2e^{-bt} \quad \Leftrightarrow$

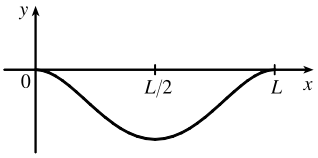
$e^{bt}e^{-at} > \dfrac{b^2}{a^2} \quad \Leftrightarrow \quad e^{(b-a)t} > \left(\dfrac{b}{a}\right)^2 \quad \Leftrightarrow \quad (b-a)t > \ln\left(\dfrac{b}{a}\right)^2 \quad \Leftrightarrow \quad t > \dfrac{2\ln(b/a)}{b-a} = 2D$, so C is CU on $(2D, \infty)$ and CD on $(0, 2D)$. The inflection point is $(2D, C(2D))$. For the graph shown, $K = 1$, $a = 1$, $b = 2$, $D = \ln 2$, $C(D) = \frac{1}{4}$, and $C(2D) = \frac{3}{16}$. The graph tells us that when the drug is injected into the bloodstream, its concentration rises rapidly to a maximum at time D, then falls, reaching its maximum rate of decrease at time $2D$, then continues to decrease more and more slowly, approaching 0 as $t \to \infty$.



63. $y = -\dfrac{W}{24EI}x^4 + \dfrac{WL}{12EI}x^3 - \dfrac{WL^2}{24EI}x^2 = -\dfrac{W}{24EI}x^2\left(x^2 - 2Lx + L^2\right)$

$= \dfrac{-W}{24EI}x^2(x-L)^2 = cx^2(x-L)^2$

where $c = -\dfrac{W}{24EI}$ is a negative constant and $0 \le x \le L$. We sketch $f(x) = cx^2(x-L)^2$ for $c = -1$. $f(0) = f(L) = 0$.



$f'(x) = cx^2[2(x-L)] + (x-L)^2(2cx) = 2cx(x-L)[x + (x-L)] = 2cx(x-L)(2x-L)$. So for $0 < x < L$, $f'(x) > 0 \quad \Leftrightarrow \quad x(x-L)(2x-L) < 0$ [since $c < 0$] $\quad \Leftrightarrow \quad L/2 < x < L$ and $f'(x) < 0 \quad \Leftrightarrow \quad 0 < x < L/2$. Thus, f is increasing on $(L/2, L)$ and decreasing on $(0, L/2)$, and there is a local and absolute minimum at the point $(L/2, f(L/2)) = \left(L/2, cL^4/16\right)$. $f'(x) = 2c[x(x-L)(2x-L)] \quad \Rightarrow$

$f''(x) = 2c[1(x-L)(2x-L) + x(1)(2x-L) + x(x-L)(2)] = 2c(6x^2 - 6Lx + L^2) = 0 \quad \Leftrightarrow$

$x = \dfrac{6L \pm \sqrt{12L^2}}{12} = \frac{1}{2}L \pm \frac{\sqrt{3}}{6}L$, and these are the x-coordinates of the two inflection points.

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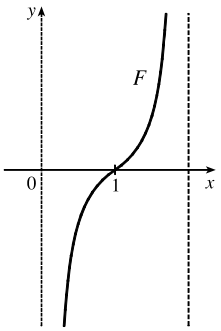
64. $F(x) = -\dfrac{k}{x^2} + \dfrac{k}{(x-2)^2}$, where $k > 0$ and $0 < x < 2$. For $0 < x < 2$, $x - 2 < 0$, so $F'(x) = \dfrac{2k}{x^3} - \dfrac{2k}{(x-2)^3} > 0$ and F is increasing. $\lim\limits_{x \to 0^+} F(x) = -\infty$ and $\lim\limits_{x \to 2^-} F(x) = \infty$, so $x = 0$ and $x = 2$ are vertical asymptotes. Notice that when the middle particle is at $x = 1$, the net force acting on it is 0. When $x > 1$, the net force is positive, meaning that it acts to the right. And if the particle approaches $x = 2$, the force on it rapidly becomes very large. When $x < 1$, the net force is negative, so it acts to the left. If the particle approaches 0, the force becomes very large to the left.



65. $y = \dfrac{x^2+1}{x+1}$. Long division gives us:

$$\begin{array}{r} x-1 \\ x+1 \overline{\big)\, x^2 \quad +1} \\ \underline{x^2 + x} \\ -x+1 \\ \underline{-x-1} \\ 2 \end{array}$$

Thus, $y = f(x) = \dfrac{x^2+1}{x+1} = x - 1 + \dfrac{2}{x+1}$ and $f(x) - (x-1) = \dfrac{2}{x+1} = \dfrac{\frac{2}{x}}{1+\frac{1}{x}}$ [for $x \neq 0$] $\to 0$ as $x \to \pm\infty$.

So the line $y = x - 1$ is a slant asymptote (SA).

66. $y = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x}$. Long division gives us:

$$\begin{array}{r} 4x+2 \\ x^2-3x \overline{\big)\, 4x^3 - 10x^2 - 11x + 1} \\ \underline{4x^3 - 12x^2} \\ 2x^2 - 11x \\ \underline{2x^2 - \ 6x} \\ -5x+1 \end{array}$$

Thus, $y = f(x) = \dfrac{4x^3 - 10x^2 - 11x + 1}{x^2 - 3x} = 4x + 2 + \dfrac{-5x+1}{x^2-3x}$ and $f(x) - (4x+2) = \dfrac{-5x+1}{x^2-3x} = \dfrac{-\frac{5}{x} + \frac{1}{x^2}}{1 - \frac{3}{x}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 4x + 2$ is a slant asymptote (SA).

67. $y = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2}$. Long division gives us:

$$\begin{array}{r} 2x-3 \\ x^2-x-2 \overline{\big)\, 2x^3 - 5x^2 + 3x} \\ \underline{2x^3 - 2x^2 - 4x} \\ -3x^2 + 7x \\ \underline{-3x^2 + 3x + 6} \\ 4x - 6 \end{array}$$

Thus, $y = f(x) = \dfrac{2x^3 - 5x^2 + 3x}{x^2 - x - 2} = 2x - 3 + \dfrac{4x-6}{x^2-x-2}$ and $f(x) - (2x-3) = \dfrac{4x-6}{x^2-x-2} = \dfrac{\frac{4}{x} - \frac{6}{x^2}}{1 - \frac{1}{x} - \frac{1}{x^2}}$

[for $x \neq 0$] $\to \frac{0}{1} = 0$ as $x \to \pm\infty$. So the line $y = 2x - 3$ is a slant asymptote (SA).

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68. $y = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x}$. Long division gives us:

$$\begin{array}{r|l} & -3x+1 \\ \hline 2x^3 - x & -6x^4 + 2x^3 \quad\quad +3 \\ & -6x^4 \quad\quad +3x^2 \\ \hline & \quad 2x^3 - 3x^2 \\ & \quad 2x^3 \quad\quad -x \\ \hline & \quad -3x^2 + x + 3 \end{array}$$

Thus, $y = f(x) = \dfrac{-6x^4 + 2x^3 + 3}{2x^3 - x} = -3x + 1 + \dfrac{-3x^2 + x + 3}{2x^3 - x}$ and

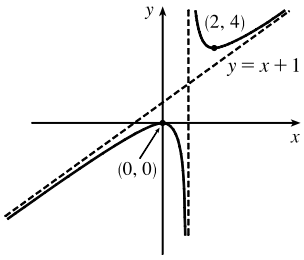
$f(x) - (-3x + 1) = \dfrac{-3x^2 + x + 3}{2x^3 - x} = \dfrac{-\dfrac{3}{x} + \dfrac{1}{x^2} + \dfrac{3}{x^3}}{2 - \dfrac{1}{x^2}}$ [for $x \neq 0$] $\to \frac{0}{2} = 0$ as $x \to \pm\infty$. So the line $y = -3x + 1$ is a slant asymptote (SA).

69. $y = f(x) = \dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$ **A.** $D = (-\infty, 1) \cup (1, \infty)$ **B.** x-intercept: $f(x) = 0 \;\Leftrightarrow\; x = 0$; y-intercept: $f(0) = 0$ **C.** No symmetry **D.** $\lim\limits_{x \to 1^-} f(x) = -\infty$ and $\lim\limits_{x \to 1^+} f(x) = \infty$, so $x = 1$ is a VA.

$\lim\limits_{x \to \pm\infty} [f(x) - (x+1)] = \lim\limits_{x \to \pm\infty} \dfrac{1}{x-1} = 0$, so the line $y = x + 1$ is a SA.

E. $f'(x) = 1 - \dfrac{1}{(x-1)^2} = \dfrac{(x-1)^2 - 1}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 1)$ and $(1, 2)$. **F.** Local maximum value $f(0) = 0$, local minimum value $f(2) = 4$ **G.** $f''(x) = \dfrac{2}{(x-1)^3} > 0$ for $x > 1$, so f is CU on $(1, \infty)$ and f is CD on $(-\infty, 1)$. No IP

H.



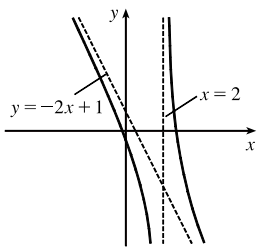
70. $y = f(x) = \dfrac{1 + 5x - 2x^2}{x - 2} = -2x + 1 + \dfrac{3}{x-2}$ **A.** $D = (-\infty, 2) \cup (2, \infty)$ **B.** x-intercepts: $f(x) = 0 \;\Leftrightarrow$ $1 + 5x - 2x^2 = 0 \;\Rightarrow\; x = \dfrac{-5 \pm \sqrt{33}}{-4} \;\Rightarrow\; x \approx -0.19, 2.69$; y-intercept: $f(0) = -\frac{1}{2}$ **C.** No symmetry

D. $\lim\limits_{x \to 2^-} f(x) = -\infty$ and $\lim\limits_{x \to 2^+} f(x) = \infty$, so $x = 2$ is a VA. $\lim\limits_{x \to \pm\infty} [f(x) - (-2x + 1)] = \lim\limits_{x \to \pm\infty} \dfrac{3}{x-2} = 0$, so $y = -2x + 1$ is a SA.

E. $f'(x) = -2 - \dfrac{3}{(x-2)^2} = \dfrac{-2(x^2 - 4x + 4) - 3}{(x-2)^2}$
$= \dfrac{-2x^2 + 8x - 11}{(x-2)^2} < 0$
for $x \neq 2$, so f is decreasing on $(-\infty, 2)$ and $(2, \infty)$. **F.** No local extrema

G. $f''(x) = \dfrac{6}{(x-2)^3} > 0$ for $x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. No IP

H.



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71. $y = f(x) = \dfrac{x^3+4}{x^2} = x + \dfrac{4}{x^2}$ **A.** $D = (-\infty, 0) \cup (0, \infty)$ **B.** x-intercept: $f(x) = 0 \Leftrightarrow x = -\sqrt[3]{4}$; no y-intercept

C. No symmetry **D.** $\lim\limits_{x\to 0} f(x) = \infty$, so $x = 0$ is a VA. $\lim\limits_{x\to\pm\infty}[f(x) - x] = \lim\limits_{x\to\pm\infty} \dfrac{4}{x^2} = 0$, so $y = x$ is a SA.

E. $f'(x) = 1 - \dfrac{8}{x^3} = \dfrac{x^3-8}{x^3} > 0$ for $x < 0$ or $x > 2$, so f is increasing on $(-\infty, 0)$ and $(2, \infty)$, and f is decreasing on $(0, 2)$. **F.** Local minimum value $f(2) = 3$, no local maximum value **G.** $f''(x) = \dfrac{24}{x^4} > 0$ for $x \neq 0$, so f is CU on $(-\infty, 0)$ and $(0, \infty)$. No IP

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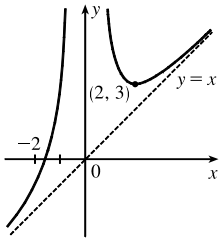

72. $y = f(x) = \dfrac{x^3}{(x+1)^2} = x - 2 + \dfrac{3x+2}{(x+1)^2}$ **A.** $D = (-\infty, -1) \cup (-1, \infty)$ **B.** x-intercept: 0; y-intercept: $f(0) = 0$

C. No symmetry **D.** $\lim\limits_{x\to -1^-} f(x) = -\infty$ and $\lim\limits_{x\to -1^+} f(x) = -\infty$, so $x = -1$ is a VA.

$\lim\limits_{x\to\pm\infty}[f(x) - (x-2)] = \lim\limits_{x\to\pm\infty} \dfrac{3x+2}{(x+1)^2} = 0$, so $y = x - 2$ is a SA.

E. $f'(x) = \dfrac{(x+1)^2(3x^2) - x^3 \cdot 2(x+1)}{[(x+1)^2]^2} = \dfrac{x^2(x+1)[3(x+1) - 2x]}{(x+1)^4} = \dfrac{x^2(x+3)}{(x+1)^3} > 0 \Leftrightarrow x < -3$ or $x > -1$ $[x \neq 0]$, so f is increasing on $(-\infty, -3)$ and $(-1, \infty)$, and f is decreasing on $(-3, -1)$.

F. Local maximum value $f(-3) = -\frac{27}{4}$, no local minimum

G. $f''(x) = \dfrac{(x+1)^3(3x^2+6x) - (x^3+3x^2)\cdot 3(x+1)^2}{[(x+1)^3]^2}$

$= \dfrac{3x(x+1)^2[(x+1)(x+2) - (x^2+3x)]}{(x+1)^6}$

$= \dfrac{3x(x^2+3x+2-x^2-3x)}{(x+1)^4} = \dfrac{6x}{(x+1)^4} > 0 \Leftrightarrow$

$x > 0$, so f is CU on $(0, \infty)$ and f is CD on $(-\infty, -1)$ and $(-1, 0)$. IP at $(0, 0)$

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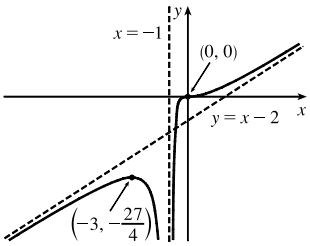

73. $y = f(x) = 1 + \frac{1}{2}x + e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$, no x-intercept [see part **F**] **C.** No symmetry

D. No VA or HA. $\lim\limits_{x\to\infty}\left[f(x) - \left(1 + \frac{1}{2}x\right)\right] = \lim\limits_{x\to\infty} e^{-x} = 0$, so $y = 1 + \frac{1}{2}x$ is a SA. **E.** $f'(x) = \frac{1}{2} - e^{-x} > 0 \Leftrightarrow$ $\frac{1}{2} > e^{-x} \Leftrightarrow -x < \ln\frac{1}{2} \Leftrightarrow x > -\ln 2^{-1} \Leftrightarrow x > \ln 2$, so f is increasing on $(\ln 2, \infty)$ and decreasing on $(-\infty, \ln 2)$. **F.** Local and absolute minimum value

$$f(\ln 2) = 1 + \tfrac{1}{2}\ln 2 + e^{-\ln 2} = 1 + \tfrac{1}{2}\ln 2 + (e^{\ln 2})^{-1}$$
$$= 1 + \tfrac{1}{2}\ln 2 + \tfrac{1}{2} = \tfrac{3}{2} + \tfrac{1}{2}\ln 2 \approx 1.85,$$

no local maximum value **G.** $f''(x) = e^{-x} > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

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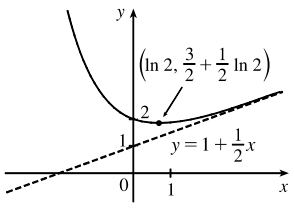

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74. $y = f(x) = 1 - x + e^{1+x/3}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 1 + e$, no x-intercept [see part **F**]

C. No symmetry **D.** No VA or HA $\lim_{x\to-\infty}[f(x) - (1-x)] = \lim_{x\to-\infty} e^{1+x/3} = 0$, so $y = 1 - x$ is a SA.

E. $f'(x) = -1 + \frac{1}{3}e^{1+x/3} > 0 \;\Leftrightarrow\; \frac{1}{3}e^{1+x/3} > 1 \;\Leftrightarrow\; e^{1+x/3} > 3 \;\Leftrightarrow\; 1 + \frac{x}{3} > \ln 3 \;\Leftrightarrow\; \frac{x}{3} > \ln 3 - 1 \;\Leftrightarrow$

$x > 3(\ln 3 - 1) \approx 0.3$, so f isincreasing on $(3\ln 3 - 3, \infty)$ and decreasing on $(-\infty, 3\ln 3 - 3)$. **F.** Local and absolute minimum value $f(3\ln 3 - 3) = 1 - (3\ln 3 - 3) + e^{1+\ln 3 - 1} = 4 - 3\ln 3 + 3 = 7 - 3\ln 3 \approx 3.7$, no local maximum value **G.** $f''(x) = \frac{1}{9}e^{1+x/3} > 0$ for all x, so f is CU on $(-\infty, \infty)$. No IP

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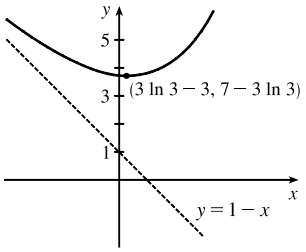


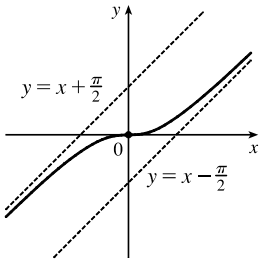
75. $y = f(x) = x - \tan^{-1} x$, $f'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{1+x^2-1}{1+x^2} = \dfrac{x^2}{1+x^2}$,

$f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x(1+x^2-x^2)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2}$.

$\lim_{x\to\infty}\left[f(x) - \left(x - \frac{\pi}{2}\right)\right] = \lim_{x\to\infty}\left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{\pi}{2} - \frac{\pi}{2} = 0$, so $y = x - \frac{\pi}{2}$ is a SA.

Also, $\lim_{x\to-\infty}\left[f(x) - \left(x + \frac{\pi}{2}\right)\right] = \lim_{x\to-\infty}\left(-\frac{\pi}{2} - \tan^{-1} x\right) = -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$,

so $y = x + \frac{\pi}{2}$ is also a SA. $f'(x) \geq 0$ for all x, with equality $\Leftrightarrow$ $x = 0$, so f is increasing on $\mathbb{R}$. $f''(x)$ has the same sign as x, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. $f(-x) = -f(x)$, so f is an odd function; its graph is symmetric about the origin. f has no local extreme values. Its only IP is at $(0, 0)$.



76. $y = f(x) = \sqrt{x^2 + 4x} = \sqrt{x(x+4)}$. $x(x+4) \geq 0 \;\Leftrightarrow\; x \leq -4$ or $x \geq 0$, so $D = (-\infty, -4] \cup [0, \infty)$.

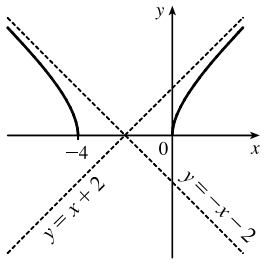
y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \;\Rightarrow\; x = -4, 0$.

$$\sqrt{x^2+4x} \mp (x+2) = \frac{\sqrt{x^2+4x} \mp (x+2)}{1} \cdot \frac{\sqrt{x^2+4x} \pm (x+2)}{\sqrt{x^2+4x} \pm (x+2)} = \frac{(x^2+4x) - (x^2+4x+4)}{\sqrt{x^2+4x} \pm (x+2)}$$
$$= \frac{-4}{\sqrt{x^2+4x} \pm (x+2)}$$

so $\lim_{x\to\pm\infty}[f(x) \mp (x+2)] = 0$. Thus, the graph of f approaches the slant asymptote $y = x + 2$ as $x \to \infty$ and it approaches the slant asymptote $y = -(x+2)$ as $x \to -\infty$. $f'(x) = \dfrac{x+2}{\sqrt{x^2+4x}}$, so $f'(x) < 0$ for $x < -4$ and $f'(x) > 0$ for $x > 0$; that is, f is decreasing on $(-\infty, -4)$ and increasing on $(0, \infty)$. There are no local extreme values. $f'(x) = (x+2)(x^2+4x)^{-1/2} \;\Rightarrow$

$$f''(x) = (x+2)\cdot\left(-\tfrac{1}{2}\right)(x^2+4x)^{-3/2}\cdot(2x+4) + (x^2+4x)^{-1/2}$$
$$= (x^2+4x)^{-3/2}\left[-(x+2)^2 + (x^2+4x)\right] = -4(x^2+4x)^{-3/2} < 0 \text{ on } D$$

so f is CD on $(-\infty, -4)$ and $(0, \infty)$. No IP



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77. $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1 \;\Rightarrow\; y=\pm\dfrac{b}{a}\sqrt{x^2-a^2}$. Now

$$\lim_{x\to\infty}\left[\frac{b}{a}\sqrt{x^2-a^2}-\frac{b}{a}x\right]=\frac{b}{a}\cdot\lim_{x\to\infty}\left(\sqrt{x^2-a^2}-x\right)\frac{\sqrt{x^2-a^2}+x}{\sqrt{x^2-a^2}+x}=\frac{b}{a}\cdot\lim_{x\to\infty}\frac{-a^2}{\sqrt{x^2-a^2}+x}=0,$$

which shows that $y=\dfrac{b}{a}x$ is a slant asymptote. Similarly,

$\lim\limits_{x\to\infty}\left[-\dfrac{b}{a}\sqrt{x^2-a^2}-\left(-\dfrac{b}{a}x\right)\right]=-\dfrac{b}{a}\cdot\lim\limits_{x\to\infty}\dfrac{-a^2}{\sqrt{x^2-a^2}+x}=0$, so $y=-\dfrac{b}{a}x$ is a slant asymptote.

78. $f(x)-x^2=\dfrac{x^3+1}{x}-x^2=\dfrac{x^3+1-x^3}{x}=\dfrac{1}{x}$, and $\lim\limits_{x\to\pm\infty}\dfrac{1}{x}=0$. Therefore, $\lim\limits_{x\to\pm\infty}[f(x)-x^2]=0$,

and so the graph of f is asymptotic to that of $y=x^2$. For purposes of differentiation, we will use $f(x)=x^2+1/x$.

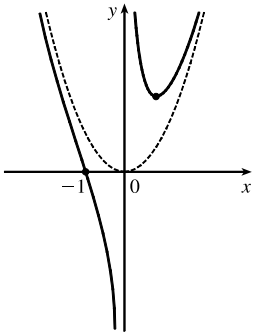
A. $D=\{x\mid x\neq 0\}$ **B.** No y-intercept; to find the x-intercept, we set $y=0 \;\Leftrightarrow\; x=-1$.

C. No symmetry **D.** $\lim\limits_{x\to0^+}\dfrac{x^3+1}{x}=\infty$ and $\lim\limits_{x\to0^-}\dfrac{x^3+1}{x}=-\infty$, so $x=0$ is a vertical asymptote. Also, the graph is asymptotic to the parabola $y=x^2$, as shown above. **E.** $f'(x)=2x-1/x^2>0 \;\Leftrightarrow\; x>\frac{1}{\sqrt[3]{2}}$, so f is increasing on $\left(\dfrac{1}{\sqrt[3]{2}},\infty\right)$ and decreasing on $(-\infty,0)$ and $\left(0,\dfrac{1}{\sqrt[3]{2}}\right)$.

F. Local minimum value $f\left(\dfrac{1}{\sqrt[3]{2}}\right)=\dfrac{3\sqrt[3]{3}}{2}$, no local maximum

G. $f''(x)=2+2/x^3>0 \;\Leftrightarrow\; x<-1$ or $x>0$, so f is CU on $(-\infty,-1)$ and $(0,\infty)$, and CD on $(-1,0)$. IP at $(-1,0)$

H.



79. $\lim\limits_{x\to\pm\infty}\left[f(x)-x^3\right]=\lim\limits_{x\to\pm\infty}\dfrac{x^4+1}{x}-\dfrac{x^4}{x}=\lim\limits_{x\to\pm\infty}\dfrac{1}{x}=0$, so the graph of f is asymptotic to that of $y=x^3$.

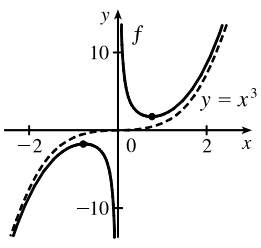
A. $D=\{x\mid x\neq 0\}$ **B.** No intercept **C.** f is symmetric about the origin. **D.** $\lim\limits_{x\to0^-}\left(x^3+\dfrac{1}{x}\right)=-\infty$ and

$\lim\limits_{x\to0^+}\left(x^3+\dfrac{1}{x}\right)=\infty$, so $x=0$ is a vertical asymptote, and as shown above, the graph of f is asymptotic to that of $y=x^3$.

E. $f'(x)=3x^2-1/x^2>0 \;\Leftrightarrow\; x^4>\frac{1}{3} \;\Leftrightarrow\; |x|>\frac{1}{\sqrt[4]{3}}$, so f is increasing on $\left(-\infty,-\dfrac{1}{\sqrt[4]{3}}\right)$ and $\left(\dfrac{1}{\sqrt[4]{3}},\infty\right)$ and decreasing on $\left(-\dfrac{1}{\sqrt[4]{3}},0\right)$ and $\left(0,\dfrac{1}{\sqrt[4]{3}}\right)$. **F.** Local maximum value $f\left(-\dfrac{1}{\sqrt[4]{3}}\right)=-4\cdot3^{-5/4}$, local minimum value $f\left(\dfrac{1}{\sqrt[4]{3}}\right)=4\cdot3^{-5/4}$

G. $f''(x)=6x+2/x^3>0 \;\Leftrightarrow\; x>0$, so f is CU on $(0,\infty)$ and CD on $(-\infty,0)$. No IP

H.



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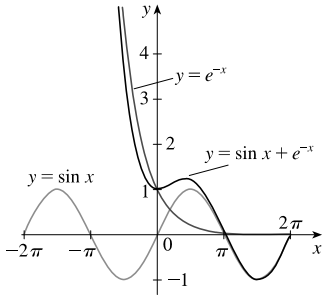
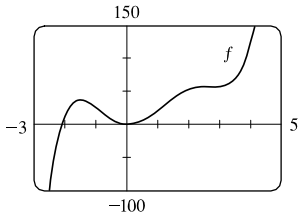
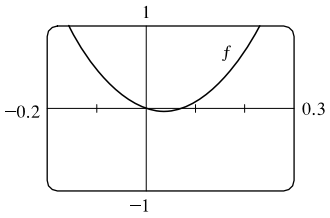
80. $f(x) = \sin x + e^{-x}$. $\lim_{x\to\infty} [f(x) - \sin x] = \lim_{x\to\infty} e^{-x} = 0$, so the graph of f is asymptotic to the graph of $\sin x$ as $x \to \infty$. $\lim_{x\to-\infty} e^{-x} = \infty$, whereas $|\sin x| \le 1$, so for large negative x, the graph of f looks like the graph of e^{-x}.

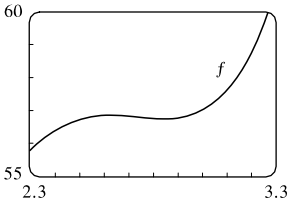


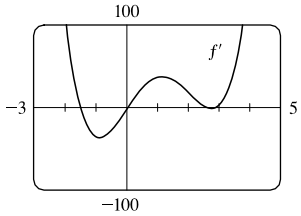
4.6 Graphing with Calculus and Technology

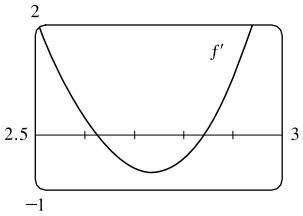
1. $f(x) = x^5 - 5x^4 - x^3 + 28x^2 - 2x \;\Rightarrow\; f'(x) = 5x^4 - 20x^3 - 3x^2 + 56x - 2 \;\Rightarrow\; f''(x) = 20x^3 - 60x^2 - 6x + 56$.
$f(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx -2.09, 0.07$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -1.50, 0.04, 2.62, 2.84$; $f''(x) = 0 \;\Leftrightarrow\; x \approx -0.89, 1.15, 2.74$.





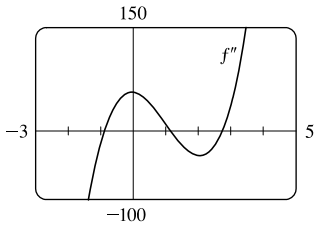






From the graphs of f', we estimate that $f' < 0$ and that f is decreasing on $(-1.50, 0.04)$ and $(2.62, 2.84)$, and that $f' > 0$ and f is increasing on $(-\infty, -1.50)$, $(0.04, 2.62)$, and $(2.84, \infty)$ with local minimum values $f(0.04) \approx -0.04$ and $f(2.84) \approx 56.73$ and local maximum values $f(-1.50) \approx 36.47$ and $f(2.62) \approx 56.83$.

From the graph of f'', we estimate that $f'' > 0$ and that f is CU on $(-0.89, 1.15)$ and $(2.74, \infty)$, and that $f'' < 0$ and f is CD on $(-\infty, -0.89)$ and $(1.15, 2.74)$. There are inflection points at about $(-0.89, 20.90)$, $(1.15, 26.57)$, and $(2.74, 56.78)$.



2. $f(x) = -2x^6 + 5x^5 + 140x^3 - 110x^2 \;\Rightarrow\; f'(x) = -12x^5 + 25x^4 + 420x^2 - 220x \;\Rightarrow\; f''(x) = -60x^4 + 100x^3 + 840x - 220$. $f(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx 0.77, 4.93$; $f'(x) = 0 \;\Leftrightarrow\; x = 0$ or

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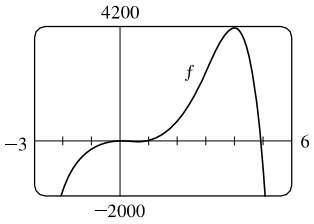
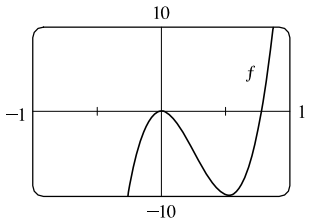
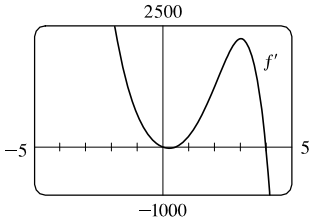
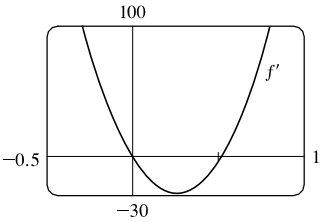
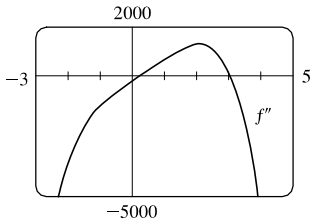
$x \approx 0.52, 3.99;\ f''(x) = 0 \quad \Leftrightarrow \quad x \approx 0.26, 3.05.$











From the graphs of f', we estimate that $f' > 0$ and that f is increasing on $(-\infty, 0)$ and $(0.52, 3.99)$, and that $f' < 0$ and that f is decreasing on $(0, 0.52)$ and $(3.99, \infty)$. f has local maximum values $f(0) = 0$ and $f(3.99) \approx 4128.20$, and f has local minimum value $f(0.52) \approx -9.91$. From the graph of f'', we estimate that $f'' > 0$ and f is CU on $(0.26, 3.05)$, and that $f'' < 0$ and f is CD on $(-\infty, 0.26)$ and $(3.05, \infty)$. There are inflection points at about $(0.26, -4.97)$ and $(3.05, 2649.46)$.

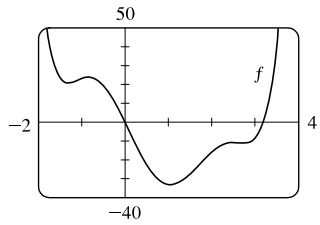
3. $f(x) = x^6 - 5x^5 + 25x^3 - 6x^2 - 48x \quad \Rightarrow$

$f'(x) = 6x^5 - 25x^4 + 75x^2 - 12x - 48 \quad \Rightarrow$

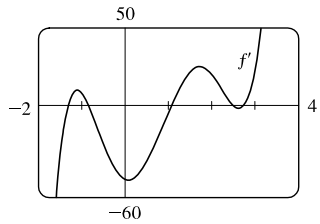
$f''(x) = 30x^4 - 100x^3 + 150x - 12.\quad f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx 3.20;$

$f'(x) = 0 \quad \Leftrightarrow \quad x \approx -1.31, -0.84, 1.06, 2.50, 2.75;\ f''(x) = 0 \quad \Leftrightarrow$

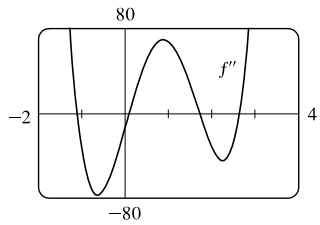
$x \approx -1.10, 0.08, 1.72, 2.64.$



From the graph of f', we estimate that f is decreasing on $(-\infty, -1.31)$, increasing on $(-1.31, -0.84)$, decreasing on $(-0.84, 1.06)$, increasing on $(1.06, 2.50)$, decreasing on $(2.50, 2.75)$, and increasing on $(2.75, \infty)$. f has local minimum values $f(-1.31) \approx 20.72$, $f(1.06) \approx -33.12$, and $f(2.75) \approx -11.33$. f has local maximum values $f(-0.84) \approx 23.71$ and $f(2.50) \approx -11.02$.



From the graph of f'', we estimate that f is CU on $(-\infty, -1.10)$, CD on $(-1.10, 0.08)$, CU on $(0.08, 1.72)$, CD on $(1.72, 2.64)$, and CU on $(2.64, \infty)$. There are inflection points at about $(-1.10, 22.09)$, $(0.08, -3.88)$, $(1.72, -22.53)$, and $(2.64, -11.18)$.



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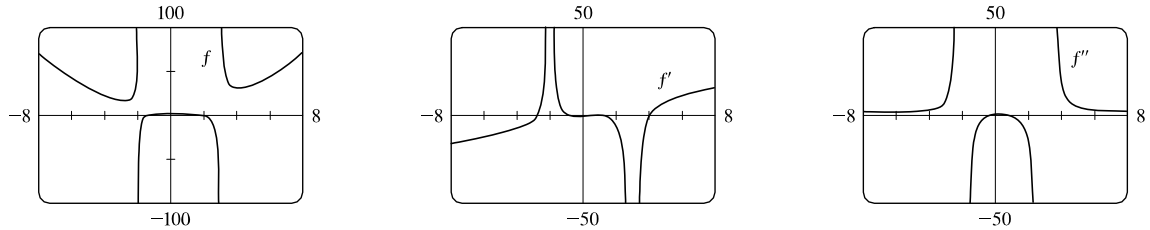
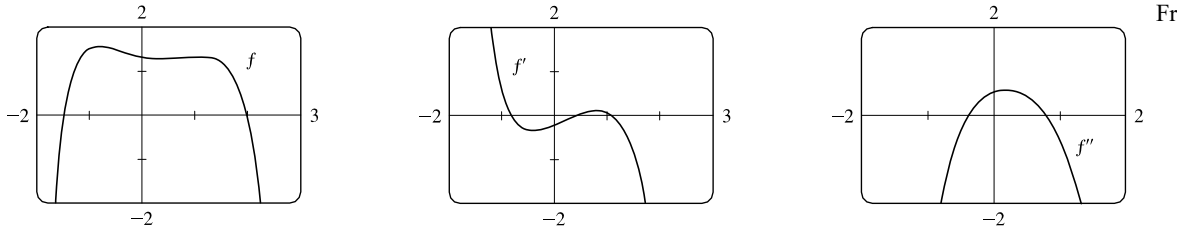
4. $f(x)=\dfrac{x^4-x^3-8}{x^2-x-6} \;\Rightarrow\; f'(x)=\dfrac{2(x^5-2x^4-11x^3+9x^2+8x-4)}{(x^2-x-6)^2} \;\Rightarrow$

$f''(x)=\dfrac{2(x^6-3x^5-15x^4+41x^3+174x^2-84x-56}{(x^2-x-6)^3}$. $f(x)=0 \;\Leftrightarrow\; x\approx -1.48$ or $x=2$; $f'(x)=0 \;\Leftrightarrow$

$x\approx -2.74,\ -0.81,\ 0.41,\ 1.08,\ 4.06$; $f''(x)=0 \;\Leftrightarrow\; x\approx -0.39,\ 0.79$. The VAs are $x=-2$ and $x=3$.

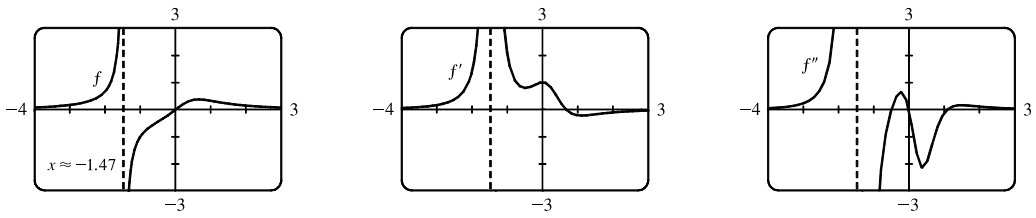




From the graphs of f', we estimate that f is decreasing on $(-\infty,-2.74)$, increasing on $(-2.74,-2)$, increasing on $(-2,-0.81)$, decreasing on $(-0.81,0.41)$, increasing on $(0.41,1.08)$, decreasing on $(1.08,3)$, decreasing on $(3,4.06)$, and increasing on $(4.06,\infty)$. f has local minimum values $f(-2.74)\approx 16.23$, $f(0.41)\approx 1.29$, and $f(4.06)\approx 30.63$.
f has local maximum values $f(-0.81)\approx 1.55$ and $f(1.08)\approx 1.34$.

From the graphs of f'', we estimate that f is CU on $(-\infty,-2)$, CD on $(-2,-0.39)$, CU on $(-0.39,0.79)$, CD on $(0.79,3)$, and CU on $(3,\infty)$. There are inflection points at about $(-0.39,1.45)$ and $(0.79,1.31)$.

5. $f(x)=\dfrac{x}{x^3+x^2+1} \;\Rightarrow\; f'(x)=-\dfrac{2x^3+x^2-1}{(x^3+x^2+1)^2} \;\Rightarrow\; f''(x)=\dfrac{2x(3x^4+3x^3+x^2-6x-3)}{(x^3+x^2+1)^3}$



From the graph of f, we see that there is a VA at $x\approx -1.47$. From the graph of f', we estimate that f is increasing on $(-\infty,-1.47)$, increasing on $(-1.47,0.66)$, and decreasing on $(0.66,\infty)$, with local maximum value $f(0.66)\approx 0.38$.

From the graph of f'', we estimate that f is CU on $(-\infty,-1.47)$, CD on $(-1.47,-0.49)$, CU on $(-0.49,0)$, CD on $(0,1.10)$, and CU on $(1.10,\infty)$. There is an inflection point at $(0,0)$ and at about $(-0.49,-0.44)$ and $(1.10,0.31)$.

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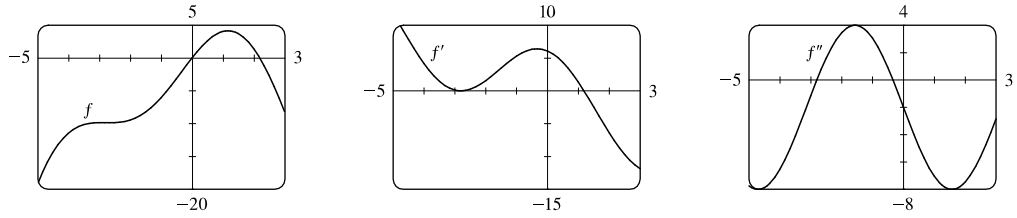
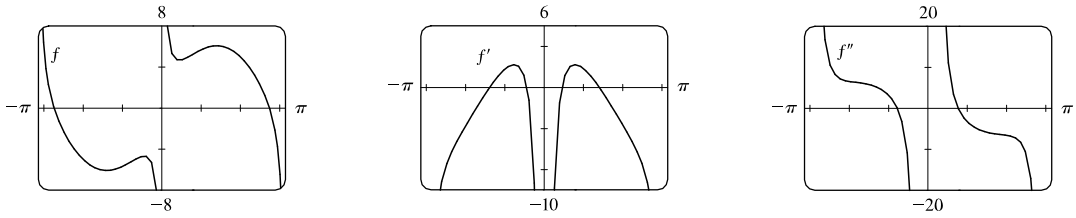
6. $f(x) = 6\sin x - x^2, -5 \le x \le 3 \quad \Rightarrow \quad f'(x) = 6\cos x - 2x \quad \Rightarrow \quad f''(x) = -6\sin x - 2$



From the graph of f', which has two negative zeros, we estimate that f is increasing on $(-5, -2.94)$, decreasing on $(-2.94, -2.66)$, increasing on $(-2.66, 1.17)$, and decreasing on $(1.17, 3)$, with local maximum values $f(-2.94) \approx -9.84$ and $f(1.17) \approx 4.16$, and local minimum value $f(-2.66) \approx -9.85$.

From the graph of f'', we estimate that f is CD on $(-5, -2.80)$, CU on $(-2.80, -0.34)$, and CD on $(-0.34, 3)$. There are inflection points at about $(-2.80, -9.85)$ and $(-0.34, -2.12)$.

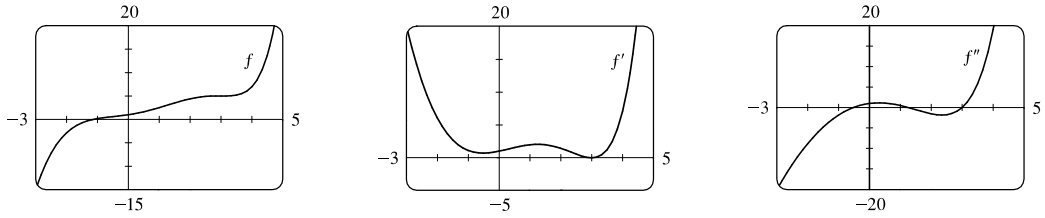
7. $f(x) = 6\sin x + \cot x, -\pi \le x \le \pi \quad \Rightarrow \quad f'(x) = 6\cos x - \csc^2 x \quad \Rightarrow \quad f''(x) = -6\sin x + 2\csc^2 x \cot x$



From the graph of f, we see that there are VAs at $x = 0$ and $x = \pm\pi$. f is an odd function, so its graph is symmetric about the origin. From the graph of f', we estimate that f is decreasing on $(-\pi, -1.40)$, increasing on $(-1.40, -0.44)$, decreasing on $(-0.44, 0)$, decreasing on $(0, 0.44)$, increasing on $(0.44, 1.40)$, and decreasing on $(1.40, \pi)$, with local minimum values $f(-1.40) \approx -6.09$ and $f(0.44) \approx 4.68$, and local maximum values $f(-0.44) \approx -4.68$ and $f(1.40) \approx 6.09$.

From the graph of f'', we estimate that f is CU on $(-\pi, -0.77)$, CD on $(-0.77, 0)$, CU on $(0, 0.77)$, and CD on $(0.77, \pi)$. There are IPs at about $(-0.77, -5.22)$ and $(0.77, 5.22)$.

8. $f(x) = e^x - 0.186x^4 \quad \Rightarrow \quad f'(x) = e^x - 0.744x^3 \quad \Rightarrow \quad f''(x) = e^x - 2.232x^2$



From the graph of f', which has two positive zeros, we estimate that f is increasing on $(-\infty, 2.973)$, decreasing on $(2.973, 3.027)$, and increasing on $(3.027, \infty)$, with local maximum value $f(2.973) \approx 5.01958$ and local minimum value $f(3.027) \approx 5.01949$.

From the graph of f'', we estimate that f is CD on $(-\infty, -0.52)$, CU on $(-0.52, 1.25)$, CD on $(1.25, 3.00)$, and CU on $(3.00, \infty)$. There are inflection points at about $(-0.52, 0.58)$, $(1.25, 3.04)$ and $(3.00, 5.01954)$.

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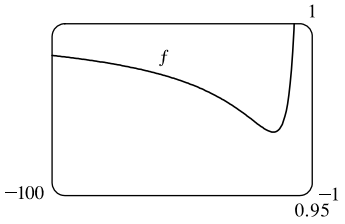
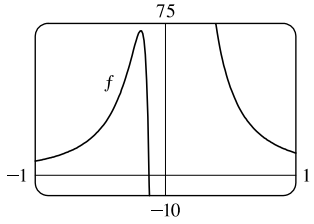
9. $f(x) = 1 + \dfrac{1}{x} + \dfrac{8}{x^2} + \dfrac{1}{x^3} \;\Rightarrow\; f'(x) = -\dfrac{1}{x^2} - \dfrac{16}{x^3} - \dfrac{3}{x^4} = -\dfrac{1}{x^4}(x^2 + 16x + 3) \;\Rightarrow$

$f''(x) = \dfrac{2}{x^3} + \dfrac{48}{x^4} + \dfrac{12}{x^5} = \dfrac{2}{x^5}(x^2 + 24x + 6)$.





From the graphs, it appears that f increases on $(-15.8, -0.2)$ and decreases on $(-\infty, -15.8)$, $(-0.2, 0)$, and $(0, \infty)$; that f has a local minimum value of $f(-15.8) \approx 0.97$ and a local maximum value of $f(-0.2) \approx 72$; that f is CD on $(-\infty, -24)$ and $(-0.25, 0)$ and is CU on $(-24, -0.25)$ and $(0, \infty)$; and that f has IPs at $(-24, 0.97)$ and $(-0.25, 60)$.

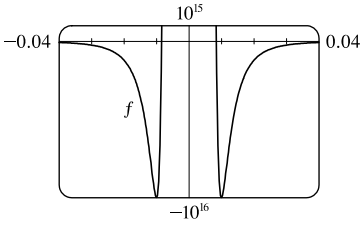
To find the exact values, note that $f' = 0 \;\Rightarrow\; x = \dfrac{-16 \pm \sqrt{256 - 12}}{2} = -8 \pm \sqrt{61}$ $[\approx -0.19$ and $-15.81]$.

f' is positive (f is increasing) on $\left(-8 - \sqrt{61}, -8 + \sqrt{61}\right)$ and f' is negative (f is decreasing) on $\left(-\infty, -8 - \sqrt{61}\right)$, $\left(-8 + \sqrt{61}, 0\right)$, and $(0, \infty)$. $f'' = 0 \;\Rightarrow\; x = \dfrac{-24 \pm \sqrt{576 - 24}}{2} = -12 \pm \sqrt{138}$ $[\approx -0.25$ and $-23.75]$. f'' is positive (f is CU) on $\left(-12 - \sqrt{138}, -12 + \sqrt{138}\right)$ and $(0, \infty)$ and f'' is negative (f is CD) on $\left(-\infty, -12 - \sqrt{138}\right)$ and $\left(-12 + \sqrt{138}, 0\right)$.

10. $f(x) = \dfrac{1}{x^8} - \dfrac{c}{x^4}$ $[c = 2 \times 10^8] \;\Rightarrow$

$f'(x) = -\dfrac{8}{x^9} + \dfrac{4c}{x^5} = -\dfrac{4}{x^9}(2 - cx^4) \;\Rightarrow$

$f''(x) = \dfrac{72}{x^{10}} - \dfrac{20c}{x^6} = \dfrac{4}{x^{10}}(18 - 5cx^4)$.



From the graph, it appears that f increases on $(-0.01, 0)$ and $(0.01, \infty)$ and decreases on $(-\infty, -0.01)$ and $(0, 0.01)$; that f has a local minimum value of $f(\pm 0.01) = -10^{16}$; and that f is CU on $(-0.012, 0)$ and $(0, 0.012)$ and f is CD on $(-\infty, -0.012)$ and $(0.012, \infty)$.

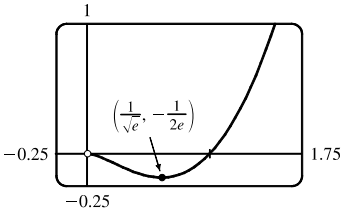
To find the exact values, note that $f' = 0 \;\Rightarrow\; x^4 = \frac{2}{c} \;\Rightarrow\; x \pm \sqrt[4]{\frac{2}{c}} = \pm\frac{1}{100}$ $[c = 2 \times 10^8]$. f' is positive (f is increasing) on $(-0.01, 0)$ and $(0.01, \infty)$ and f' is negative (f is decreasing) on $(-\infty, -0.01)$ and $(0, 0.01)$.

$f'' = 0 \;\Rightarrow\; x^4 = \dfrac{18}{5c} \;\Rightarrow\; x = \pm\sqrt[4]{\dfrac{18}{5c}} = \pm\dfrac{1}{100}\sqrt[4]{1.8}$ $[\approx \pm 0.0116]$. f'' is positive (f is CU) on $\left(-\frac{1}{100}\sqrt[4]{1.8}, 0\right)$ and $\left(0, \frac{1}{100}\sqrt[4]{1.8}\right)$ and f'' is negative (f is CD) on $\left(-\infty, -\frac{1}{100}\sqrt[4]{1.8}\right)$ and $\left(\frac{1}{100}\sqrt[4]{1.8}, \infty\right)$.

11. (a) $f(x) = x^2 \ln x$. The domain of f is $(0, \infty)$.

(b) $\lim\limits_{x \to 0^+} x^2 \ln x = \lim\limits_{x \to 0^+} \dfrac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim\limits_{x \to 0^+} \dfrac{1/x}{-2/x^3} = \lim\limits_{x \to 0^+} \left(-\dfrac{x^2}{2}\right) = 0.$

There is a hole at $(0, 0)$.



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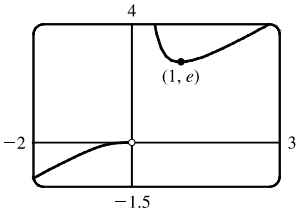
(c) It appears that there is an IP at about $(0.2, -0.06)$ and a local minimum at $(0.6, -0.18)$. $f(x) = x^2 \ln x \quad \Rightarrow$

$f'(x) = x^2(1/x) + (\ln x)(2x) = x(2\ln x + 1) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$, so f is increasing on $\left(1/\sqrt{e}, \infty\right)$, decreasing on $\left(0, 1/\sqrt{e}\right)$. By the FDT, $f\left(1/\sqrt{e}\right) = -1/(2e)$ is a local minimum value. This point is approximately $(0.6065, -0.1839)$, which agrees with our estimate.

$f''(x) = x(2/x) + (2\ln x + 1) = 2\ln x + 3 > 0 \quad \Leftrightarrow \quad \ln x > -\frac{3}{2} \quad \Leftrightarrow \quad x > e^{-3/2}$, so f is CU on $(e^{-3/2}, \infty)$ and CD on $(0, e^{-3/2})$. IP is $(e^{-3/2}, -3/(2e^3)) \approx (0.2231, -0.0747)$.

12. (a) $f(x) = xe^{1/x}$. The domain of f is $(-\infty, 0) \cup (0, \infty)$.



(b) $\lim_{x\to 0^+} xe^{1/x} = \lim_{x\to 0^+} \dfrac{e^{1/x}}{1/x} \overset{\text{H}}{=} \lim_{x\to 0^+} \dfrac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty$,

so $x = 0$ is a VA.

Also $\lim_{x\to 0^-} xe^{1/x} = 0$ since $1/x \to -\infty \quad \Rightarrow \quad e^{1/x} \to 0$.

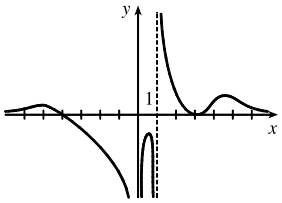
(c) It appears that there is a local minimum at $(1, 2.7)$. There are no IP and f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$.

$f(x) = xe^{1/x} \quad \Rightarrow \quad f'(x) = xe^{1/x}\left(-\dfrac{1}{x^2}\right) + e^{1/x} = e^{1/x}\left(1 - \dfrac{1}{x}\right) > 0 \quad \Leftrightarrow \quad \dfrac{1}{x} < 1 \quad \Leftrightarrow \quad x < 0 \text{ or } x > 1,$

so f is increasing on $(-\infty, 0)$ and $(1, \infty)$, and decreasing on $(0, 1)$. By the FDT, $f(1) = e$ is a local minimum value, which agrees with our estimate.

$f''(x) = e^{1/x}(1/x^2) + (1 - 1/x)e^{1/x}(-1/x^2) = (e^{1/x}/x^2)(1 - 1 + 1/x) = e^{1/x}/x^3 > 0 \quad \Leftrightarrow \quad x > 0$, so f is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. No IP

13.



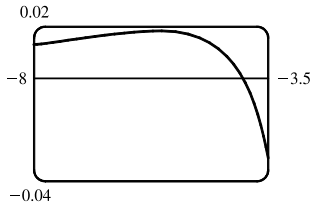
$f(x) = \dfrac{(x+4)(x-3)^2}{x^4(x-1)}$ has VA at $x = 0$ and at $x = 1$ since $\lim_{x\to 0} f(x) = -\infty$,

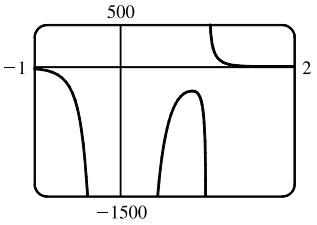
$\lim_{x\to 1^-} f(x) = -\infty$ and $\lim_{x\to 1^+} f(x) = \infty$.

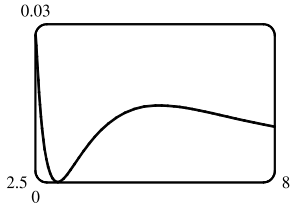
$f(x) = \dfrac{\dfrac{x+4}{x} \cdot \dfrac{(x-3)^2}{x^2}}{\dfrac{x^4}{x^3} \cdot (x-1)} \quad \left[\begin{array}{c}\text{dividing numerator}\\ \text{and denominator by } x^3\end{array}\right] = \dfrac{(1 + 4/x)(1 - 3/x)^2}{x(x-1)} \to 0$

as $x \to \pm\infty$, so f is asymptotic to the x-axis.

Since f is undefined at $x = 0$, it has no y-intercept. $f(x) = 0 \quad \Rightarrow \quad (x+4)(x-3)^2 = 0 \quad \Rightarrow \quad x = -4$ or $x = 3$, so f has x-intercepts -4 and 3. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = 3$, since f is positive as $x \to 3^-$ and as $x \to 3^+$.







From these graphs, it appears that f has three maximum values and one minimum value. The maximum values are

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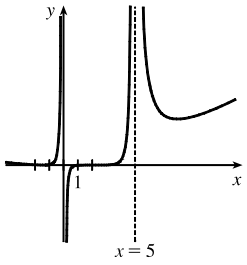
approximately $f(-5.6) = 0.0182$, $f(0.82) = -281.5$ and $f(5.2) = 0.0145$ and we know (since the graph is tangent to the x-axis at $x = 3$) that the minimum value is $f(3) = 0$.

14.

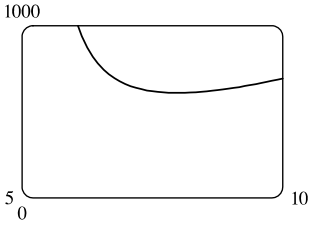


$f(x) = \dfrac{(2x+3)^2(x-2)^5}{x^3(x-5)^2}$ has VAs at $x = 0$ and $x = 5$ since $\lim\limits_{x\to 0^-} f(x) = \infty$, $\lim\limits_{x\to 0^+} f(x) = -\infty$, and $\lim\limits_{x\to 5} f(x) = \infty$. No HA since $\lim\limits_{x\to\pm\infty} f(x) = \infty$.

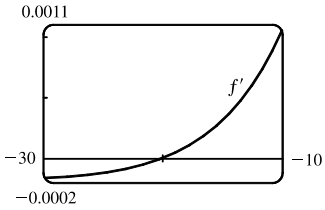
Since f is undefined at $x = 0$, it has no y-intercept.

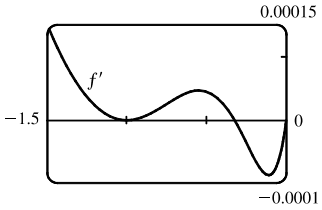
$f(x) = 0 \quad \Leftrightarrow \quad (2x+3)^2\,(x-2)^5 = 0 \quad \Leftrightarrow \quad x = -\frac{3}{2}$ or $x = 2$, so f has x-intercepts at $-\frac{3}{2}$ and 2. Note, however, that the graph of f is only tangent to the x-axis and does not cross it at $x = -\frac{3}{2}$, since f is positive as $x \to \left(-\frac{3}{2}\right)^-$ and as $x \to \left(-\frac{3}{2}\right)^+$. There is a local minimum value of $f\left(-\frac{3}{2}\right) = 0$.

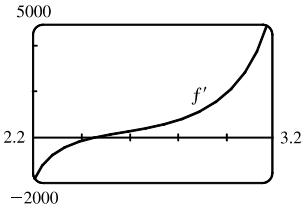
The only "mystery" feature is the local minimum to the right of the VA $x = 5$. From the graph, we see that $f(7.98) \approx 609$ is a local minimum value.



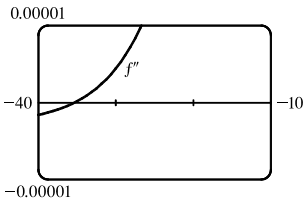
15. $f(x) = \dfrac{x^2(x+1)^3}{(x-2)^2(x-4)^4} \quad \Rightarrow \quad f'(x) = -\dfrac{x(x+1)^2(x^3+18x^2-44x-16)}{(x-2)^3(x-4)^5}$ [from CAS].

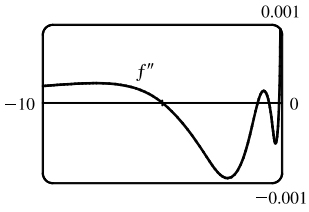


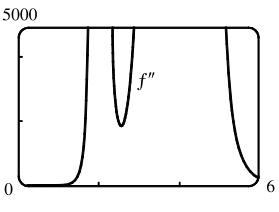




From the graphs of f', it seems that the critical points which indicate extrema occur at $x \approx -20$, -0.3, and 2.5, as estimated in Example 3. (There is another critical point at $x = -1$, but the sign of f' does not change there.) We differentiate again, obtaining $f''(x) = 2\dfrac{(x+1)(x^6+36x^5+6x^4-628x^3+684x^2+672x+64)}{(x-2)^4(x-4)^6}$.







From the graphs of f'', it appears that f is CU on $(-35.3, -5.0)$, $(-1, -0.5)$, $(-0.1, 2)$, $(2, 4)$, and $(4, \infty)$ and CD on $(-\infty, -35.3)$, $(-5.0, -1)$ and $(-0.5, -0.1)$. We check back on the graphs of f to find the y-coordinates of the inflection points, and find that these points are approximately $(-35.3, -0.015)$, $(-5.0, -0.005)$, $(-1, 0)$, $(-0.5, 0.00001)$, and $(-0.1, 0.0000066)$.

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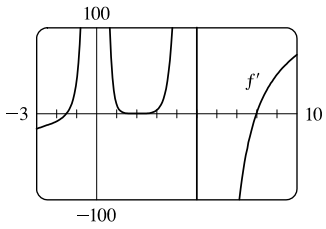
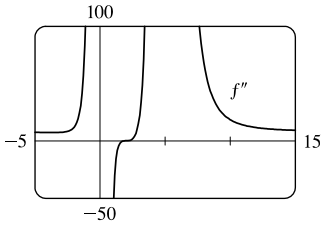
16. From a CAS, $$f'(x) = \frac{2(x-2)^4(2x+3)(2x^3-14x^2-10x-45)}{x^4(x-5)^3}$$

and $$f''(x) = \frac{2(x-2)^3(4x^6-56x^5+216x^4+460x^3+805x^2+1710x+5400)}{x^5(x-5)^4}$$



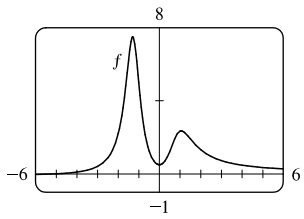
From Exercise 14 and $f'(x)$ above, we know that the zeros of f' are -1.5, 2, and 7.98. From the graph of f', we conclude that f is decreasing on $(-\infty, -1.5)$, increasing on $(-1.5, 0)$ and $(0, 5)$, decreasing on $(5, 7.98)$, and increasing on $(7.98, \infty)$.

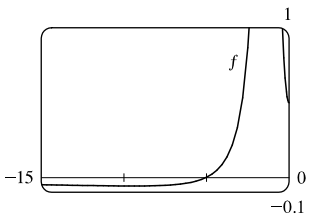


From $f''(x)$, we know that $x = 2$ is a zero, and the graph of f'' shows us that $x = 2$ is the only zero of f''. Thus, f is CU on $(-\infty, 0)$, CD on $(0, 2)$, CU on $(2, 5)$, and CU on $(5, \infty)$.

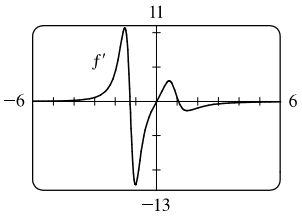
17. $f(x) = \dfrac{x^3+5x^2+1}{x^4+x^3-x^2+2}$. From a CAS, $f'(x) = \dfrac{-x(x^5+10x^4+6x^3+4x^2-3x-22)}{(x^4+x^3-x^2+2)^2}$ and

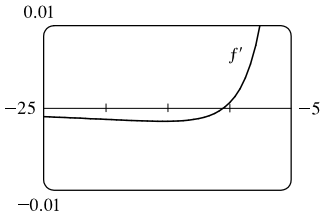
$$f''(x) = \frac{2(x^9+15x^8+18x^7+21x^6-9x^5-135x^4-76x^3+21x^2+6x+22)}{(x^4+x^3-x^2+2)^3}$$





The first graph of f shows that $y = 0$ is a HA. As $x \to \infty$, $f(x) \to 0$ through positive values. As $x \to -\infty$, it is not clear if $f(x) \to 0$ through positive or negative values. The second graph of f shows that f has an x-intercept near -5, and will have a local minimum and inflection point to the left of -5.





From the two graphs of f', we see that f' has four zeros. We conclude that f is decreasing on $(-\infty, -9.41)$, increasing on $(-9.41, -1.29)$, decreasing on $(-1.29, 0)$, increasing on $(0, 1.05)$, and decreasing on $(1.05, \infty)$. We have local minimum values $f(-9.41) \approx -0.056$ and $f(0) = 0.5$, and local maximum values $f(-1.29) \approx 7.49$ and $f(1.05) \approx 2.35$.

[continued]

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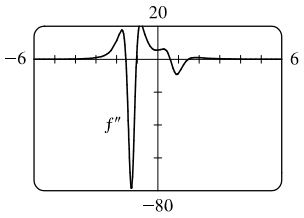
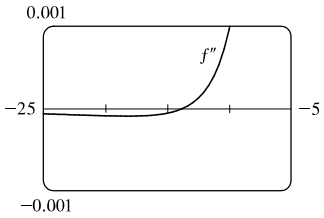
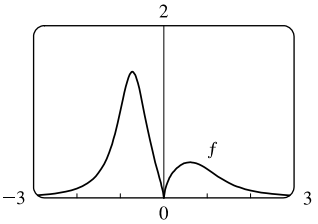
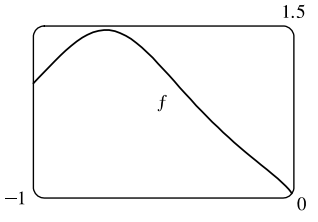


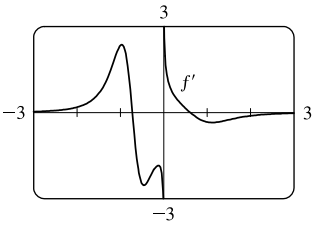


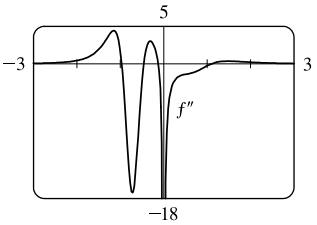
From the two graphs of f'', we see that f'' has five zeros. We conclude that f is CD on $(-\infty, -13.81)$, CU on $(-13.81, -1.55)$, CD on $(-1.55, -1.03)$, CU on $(-1.03, 0.60)$, CD on $(0.60, 1.48)$, and CU on $(1.48, \infty)$. There are five inflection points: $(-13.81, -0.05)$, $(-1.55, 5.64)$, $(-1.03, 5.39)$, $(0.60, 1.52)$, and $(1.48, 1.93)$.

18. $y = f(x) = \dfrac{x^{2/3}}{1 + x + x^4}$. From a CAS, $y' = -\dfrac{10x^4 + x - 2}{3x^{1/3}(x^4 + x + 1)^2}$ and $y'' = \dfrac{2(65x^8 - 14x^5 - 80x^4 + 2x^2 - 8x - 1)}{9x^{4/3}(x^4 + x + 1)^3}$







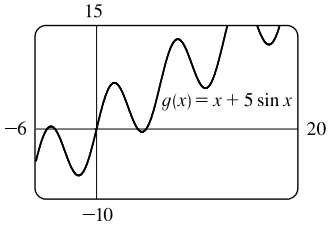


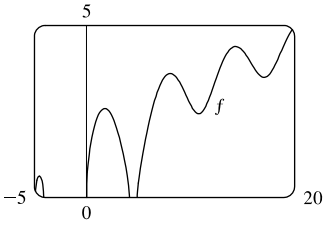
$f'(x)$ does not exist at $x = 0$ and $f'(x) = 0 \quad \Leftrightarrow \quad x \approx -0.72$ and 0.61, so f is increasing on $(-\infty, -0.72)$, decreasing on $(-0.72, 0)$, increasing on $(0, 0.61)$, and decreasing on $(0.61, \infty)$. There is a local maximum value of $f(-0.72) \approx 1.46$ and a local minimum value of $f(0.61) \approx 0.41$. $f''(x)$ does not exist at $x = 0$ and $f''(x) = 0 \quad \Leftrightarrow \quad x \approx -0.97, -0.46, -0.12,$ and 1.11, so f is CU on $(-\infty, -0.97)$, CD on $(-0.97, -0.46)$, CU on $(-0.46, -0.12)$, CD on $(-0.12, 0)$, CD on $(0, 1.11)$, and CU on $(1.11, \infty)$. There are inflection points at $(-0.97, 1.08)$, $(-0.46, 1.01)$, $(-0.12, 0.28)$, and $(1.11, 0.29)$.

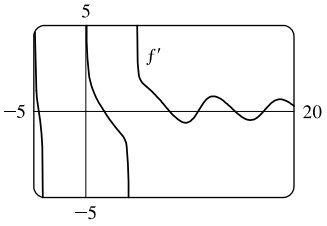
19. $y = f(x) = \sqrt{x + 5\sin x},\ x \le 20$.

From a CAS, $y' = \dfrac{5\cos x + 1}{2\sqrt{x + 5\sin x}}$ and $y'' = -\dfrac{10\cos x + 25\sin^2 x + 10x\sin x + 26}{4(x + 5\sin x)^{3/2}}$.

We'll start with a graph of $g(x) = x + 5\sin x$. Note that $f(x) = \sqrt{g(x)}$ is only defined if $g(x) \ge 0$. $g(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x \approx -4.91, -4.10, 4.10,$ and 4.91. Thus, the domain of f is $[-4.91, -4.10] \cup [0, 4.10] \cup [4.91, 20]$.







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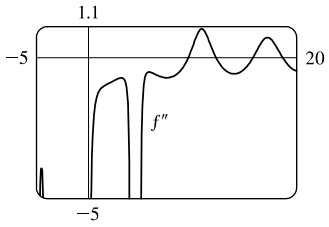
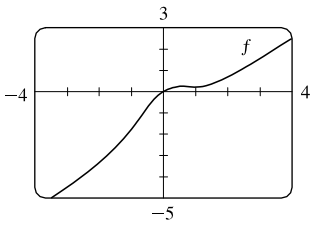
From the expression for y', we see that $y' = 0 \quad \Leftrightarrow \quad 5\cos x + 1 = 0 \quad \Rightarrow \quad x_1 = \cos^{-1}\left(-\frac{1}{5}\right) \approx 1.77$ and $x_2 = 2\pi - x_1 \approx -4.51$ (not in the domain of f). The leftmost zero of f' is $x_1 - 2\pi \approx -4.51$. Moving to the right, the zeros of f' are x_1, $x_1 + 2\pi$, $x_2 + 2\pi$, $x_1 + 4\pi$, and $x_2 + 4\pi$. Thus, f is increasing on $(-4.91, -4.51)$, decreasing on $(-4.51, -4.10)$, increasing on $(0, 1.77)$, decreasing on $(1.77, 4.10)$, increasing on $(4.91, 8.06)$, decreasing on $(8.06, 10.79)$, increasing on $(10.79, 14.34)$, decreasing on $(14.34, 17.08)$, and increasing on $(17.08, 20)$. The local maximum values are $f(-4.51) \approx 0.62$, $f(1.77) \approx 2.58$, $f(8.06) \approx 3.60$, and $f(14.34) \approx 4.39$. The local minimum values are $f(10.79) \approx 2.43$ and $f(17.08) \approx 3.49$.

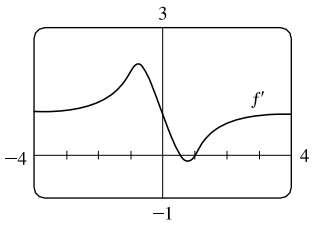
f is CD on $(-4.91, -4.10)$, $(0, 4.10)$, $(4.91, 9.60)$, CU on $(9.60, 12.25)$, CD on $(12.25, 15.81)$, CU on $(15.81, 18.65)$, and CD on $(18.65, 20)$. There are inflection points at $(9.60, 2.95)$, $(12.25, 3.27)$, $(15.81, 3.91)$, and $(18.65, 4.20)$.

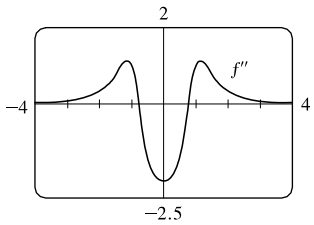


20. $y = f(x) = x - \tan^{-1} x^2$. From a CAS, $y' = \dfrac{x^4 - 2x + 1}{x^4 + 1}$ and $y'' = \dfrac{2(3x^4 - 1)}{(x^4 + 1)^2}$. $y' = 0 \quad \Leftrightarrow \quad x \approx 0.54$ or $x = 1$.

$y'' = 0 \quad \Leftrightarrow \quad x \approx \pm 0.76$.



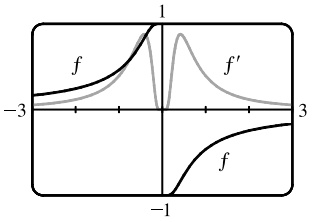


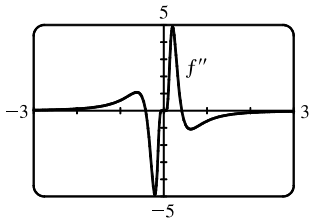


From the graphs of f and f', we estimate that f is increasing on $(-\infty, 0.54)$, decreasing on $(0.54, 1)$, and increasing on $(1, \infty)$. f has local maximum value $f(0.54) \approx 0.26$ and local minimum value $f(1) \approx 0.21$.

From the graph of f'', we estimate that f is CU on $(-\infty, -0.76)$, CD on $(-0.76, 0.76)$, and CU on $(0.76, \infty)$. There are inflection points at about $(-0.76, -1.28)$ and $(0.76, 0.24)$.

21. $y = f(x) = \dfrac{1 - e^{1/x}}{1 + e^{1/x}}$. From a CAS, $y' = \dfrac{2e^{1/x}}{x^2(1 + e^{1/x})^2}$ and $y'' = \dfrac{-2e^{1/x}(1 - e^{1/x} + 2x + 2xe^{1/x})}{x^4(1 + e^{1/x})^3}$.





f is an odd function defined on $(-\infty, 0) \cup (0, \infty)$. Its graph has no x- or y-intercepts. Since $\lim_{x \to \pm\infty} f(x) = 0$, the x-axis is a HA. $f'(x) > 0$ for $x \neq 0$, so f is increasing on $(-\infty, 0)$ and $(0, \infty)$. It has no local extreme values.

$f''(x) = 0$ for $x \approx \pm 0.417$, so f is CU on $(-\infty, -0.417)$, CD on $(-0.417, 0)$, CU on $(0, 0.417)$, and CD on $(0.417, \infty)$. f has IPs at $(-0.417, 0.834)$ and $(0.417, -0.834)$.

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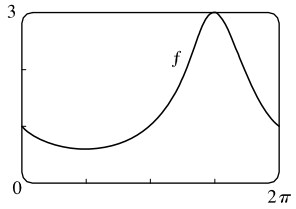
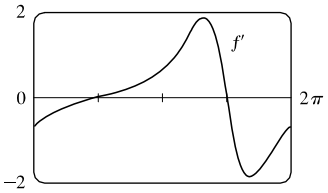
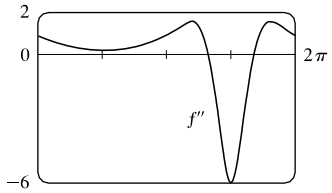
22. $y = f(x) = \dfrac{3}{3 + 2\sin x}$. From a CAS, $y' = -\dfrac{6\cos x}{(3 + 2\sin x)^2}$ and $y'' = \dfrac{6(2\sin^2 x + 4\cos^2 x + 3\sin x)}{(3 + 2\sin x)^3}$. Since f is periodic with period 2π, we'll restrict our attention to the interval $[0, 2\pi)$. $y' = 0 \;\Leftrightarrow\; 6\cos x = 0 \;\Leftrightarrow\; x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$. $y'' = 0 \;\Leftrightarrow\; x \approx 4.16$ or 5.27.





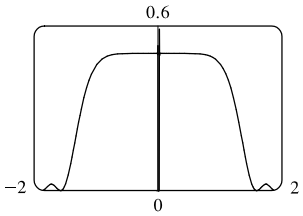


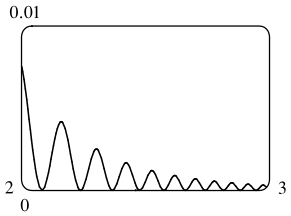
From the graphs of f and f', we conclude that f is decreasing on $\left(0, \frac{\pi}{2}\right)$, increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, and decreasing on $\left(\frac{3\pi}{2}, 2\pi\right)$. f has local minimum value $f\left(\frac{\pi}{2}\right) = \frac{3}{5}$ and local maximum value $f\left(\frac{3\pi}{2}\right) = 3$.

From the graph of f'', we conclude that f is CU on $(0, 4.16)$, CD on $(4.16, 5.27)$, and CU on $(5.27, 2\pi)$. There are inflection points at about $(4.16, 2.31)$ and $(5.27, 2.31)$.

23. $f(x) = \dfrac{1 - \cos(x^4)}{x^8} \geq 0$. f is an even function, so its graph is symmetric with respect to the y-axis. The first graph shows that f levels off at $y = \frac{1}{2}$ for $|x| < 0.7$. It also shows that f then drops to the x-axis. Your graphing utility may show some severe oscillations near the origin, but there are none. See the discussion in Section 2.2 after Example 2, as well as "Lies My Calculator and Computer Told Me" on the website.

The second graph indicates that as $|x|$ increases, f has progressively smaller humps.

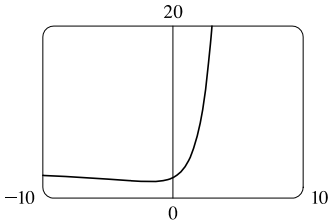


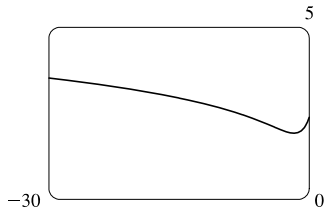


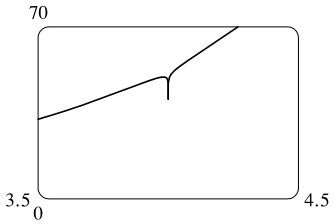
24. $f(x) = e^x + \ln|x - 4|$. The first graph shows the big picture of f but conceals hidden behavior.

The second graph shows that for large negative values of x, f looks like $g(x) = \ln|x|$. It also shows a minimum value and a point of inflection.

The third graph hints at the vertical asymptote that we know exists at $x = 4$ because $\lim\limits_{x \to 4} (e^x + \ln|x - 4|) = -\infty$.







A graphing calculator is unable to show much of the dip of the curve toward the vertical asymptote because of limited resolution. A computer can show more if we restrict ourselves to a narrow interval around $x = 4$. See the solution to Exercise 2.2.46 for a hand-drawn graph of this function.

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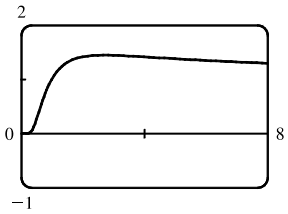
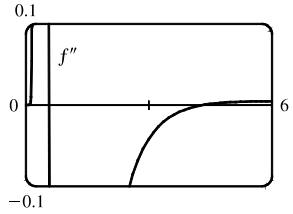
25. (a) $f(x) = x^{1/x}$



(b) Recall that $a^b = e^{b \ln a}$. $\lim_{x \to 0^+} x^{1/x} = \lim_{x \to 0^+} e^{(1/x) \ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x) \ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim_{x \to \infty} x^{1/x} = \lim_{x \to \infty} e^{(1/x) \ln x}$, but $\lim_{x \to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim_{x \to \infty} \dfrac{1/x}{1} = 0$, so $\lim_{x \to \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

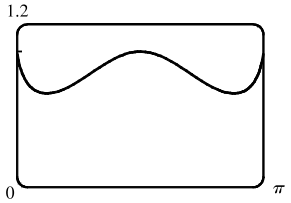
(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \quad \Rightarrow \quad y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \quad \Rightarrow$ $\ln x = 1 \quad \Rightarrow \quad x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

26. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n + 1)\pi)$, so we have graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln \sin x$, so

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \sin x \ln \sin x = \lim_{x \to 0^+} \frac{\ln \sin x}{\csc x} \overset{\text{H}}{=} \lim_{x \to 0^+} \frac{\cot x}{-\csc x \cot x}$$
$$= \lim_{x \to 0^+} (-\sin x) = 0 \quad \Rightarrow \quad \lim_{x \to 0^+} y = e^0 = 1.$$

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$.

$y = (\sin x)^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln \sin x \quad \Rightarrow \quad \dfrac{y'}{y} = (\sin x)\left(\dfrac{\cos x}{\sin x}\right) + (\ln \sin x)\cos x = \cos x\,(1 + \ln \sin x) \quad \Rightarrow$ $y' = (\sin x)^{\sin x}(\cos x)(1 + \ln \sin x)$. $y' = 0 \quad \Rightarrow \quad \cos x = 0$ or $\ln \sin x = -1 \quad \Rightarrow \quad x_2 = \frac{\pi}{2}$ or $\sin x = e^{-1}$. On $(0, \pi)$, $\sin x = e^{-1} \quad \Rightarrow \quad x_1 = \sin^{-1}(e^{-1})$ and $x_3 = \pi - \sin^{-1}(e^{-1})$. Approximating these points gives us $(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_2, f(x_2)) \approx (1.5708, 1)$, and $(x_3, f(x_3)) \approx (2.7649, 0.6922)$. The approximations confirm our estimates.

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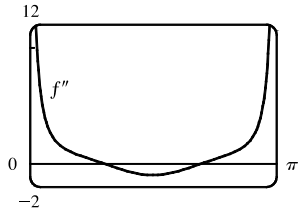
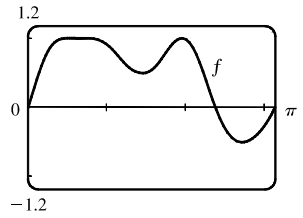
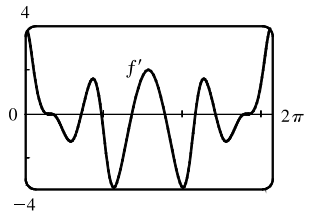
(d)

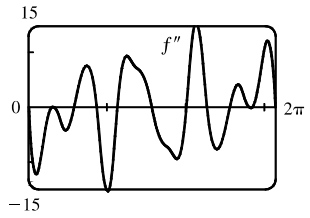


From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

27.



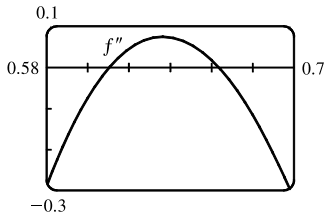


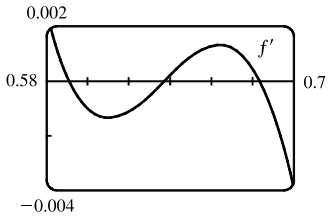


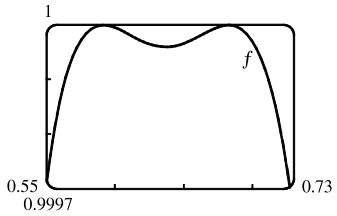
From the graph of $f(x) = \sin(x + \sin 3x)$ in the viewing rectangle $[0, \pi]$ by $[-1.2, 1.2]$, it looks like f has two maxima and two minima. If we calculate and graph $f'(x) = [\cos(x + \sin 3x)]\,(1 + 3\cos 3x)$ on $[0, 2\pi]$, we see that the graph of f' appears to be almost tangent to the x-axis at about $x = 0.7$. The graph of

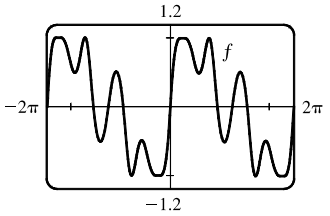
$$f'' = -\,[\sin(x + \sin 3x)]\,(1 + 3\cos 3x)^2 + \cos(x + \sin 3x)(-9\sin 3x)$$

is even more interesting near this x-value: it seems to just touch the x-axis.









If we zoom in on this place on the graph of f'', we see that f'' actually does cross the axis twice near $x = 0.65$, indicating a change in concavity for a very short interval. If we look at the graph of f' on the same interval, we see that it changes sign three times near $x = 0.65$, indicating that what we had thought was a broad extremum at about $x = 0.7$ actually consists of three extrema (two maxima and a minimum). These maximum values are roughly $f(0.59) = 1$ and $f(0.68) = 1$, and the minimum value is roughly $f(0.64) = 0.99996$. There are also a maximum value of about $f(1.96) = 1$ and minimum values of about $f(1.46) = 0.49$ and $f(2.73) = -0.51$. The points of inflection on $(0, \pi)$ are about $(0.61, 0.99998)$, $(0.66, 0.99998)$, $(1.17, 0.72)$, $(1.75, 0.77)$, and $(2.28, 0.34)$. On $(\pi, 2\pi)$, they are about $(4.01, -0.34)$, $(4.54, -0.77)$, $(5.11, -0.72)$, $(5.62, -0.99998)$, and $(5.67, -0.99998)$. There are also IP at $(0, 0)$ and $(\pi, 0)$. Note that the function is odd and periodic with period 2π, and it is also rotationally symmetric about all points of the form $((2n + 1)\pi, 0)$, n an integer.

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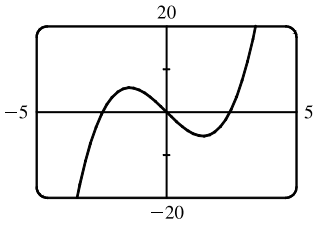
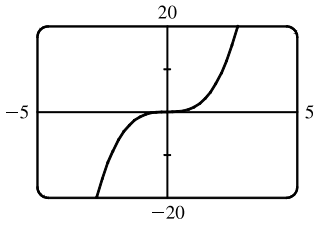
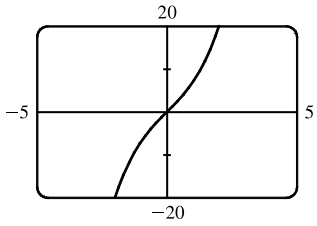
28. $f(x) = x^3 + cx = x(x^2 + c) \quad \Rightarrow \quad f'(x) = 3x^2 + c \quad \Rightarrow \quad f''(x) = 6x$



$c = -6$



$c = 0$



$c = 6$

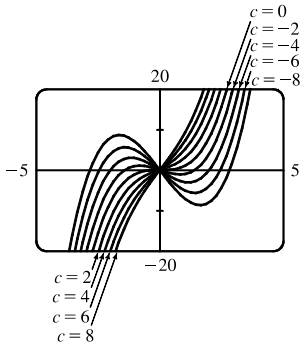
x-intercepts: When $c \geq 0$, 0 is the only x-intercept. When $c < 0$, the x-intercepts are 0 and $\pm\sqrt{-c}$.

y-intercept $= f(0) = 0$. f is odd, so the graph is symmetric with respect to the origin. $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$, so f is CD on $(-\infty, 0)$ and CU on $(0, \infty)$. The origin is the only inflection point.

If $c > 0$, then $f'(x) > 0$ for all x, so f is increasing and has no local maximum or minimum.

If $c = 0$, then $f'(x) \geq 0$ with equality at $x = 0$, so again f is increasing and has no local maximum or minimum.

If $c < 0$, then $f'(x) = 3[x^2 - (-c/3)] = 3\left(x + \sqrt{-c/3}\right)\left(x - \sqrt{-c/3}\right)$, so $f'(x) > 0$ on $\left(-\infty, -\sqrt{-c/3}\right)$ and $\left(\sqrt{-c/3}, \infty\right)$; $f'(x) < 0$ on $\left(-\sqrt{-c/3}, \sqrt{-c/3}\right)$. It follows that $f\left(-\sqrt{-c/3}\right) = -\frac{2}{3}c\sqrt{-c/3}$ is a local maximum value and $f\left(\sqrt{-c/3}\right) = \frac{2}{3}c\sqrt{-c/3}$ is a local minimum value. As c decreases (toward more negative values), the local maximum and minimum move further apart.



There is no absolute maximum or minimum value. The only transitional value of c corresponding to a change in character of the graph is $c = 0$.

29. $f(x) = x^2 + 6x + c/x \quad \Rightarrow \quad f'(x) = 2x + 6 - c/x^2 \quad \Rightarrow \quad f''(x) = 2 + 2c/x^3$

$\boldsymbol{c = 0}$: The graph is the parabola $y = x^2 + 6x$, which has x-intercepts -6 and 0, vertex $(-3, -9)$, and opens upward.

$\boldsymbol{c \neq 0}$: The parabola $y = x^2 + 6x$ is an asymptote that the graph of f approaches as $x \to \pm\infty$. The y-axis is a vertical asymptote.

$\boldsymbol{c < 0}$**:** The x-intercepts are found by solving $f(x) = 0 \quad \Leftrightarrow \quad x^3 + 6x^2 + c = g(x) = 0$. Now $g'(x) = 0 \quad \Leftrightarrow \quad x = -4$ or 0, and g (not f) has a local maximum at $x = -4$. $\quad g(-4) = 32 + c$, so if $c < -32$, the maximum is negative and there are no negative x-intercepts; if $c = -32$, the maximum is 0 and there is one negative x-intercept; if $-32 < c < 0$, the maximum is positive and there are two negative x-intercepts. In all cases, there is one positive x-intercept.

As $c \to 0^-$, the local minimum point moves down and right, approaching $(-3, -9)$. [Note that since $f'(x) = \dfrac{2x^3 + 6x^2 - c}{x^2}$, Descartes' Rule of Signs implies that f' has no positive solutions and one negative solution when $c < 0$. $\quad f''(x) = \dfrac{2(x^3 + c)}{x^3} > 0$ at that negative solution, so that critical point yields a local minimum value. This tells us that there are no local maximums when $c < 0$.] $\quad f'(x) > 0$ for $x > 0$, so f is increasing on $(0, \infty)$. From

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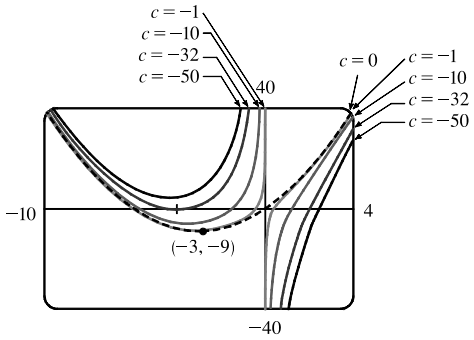
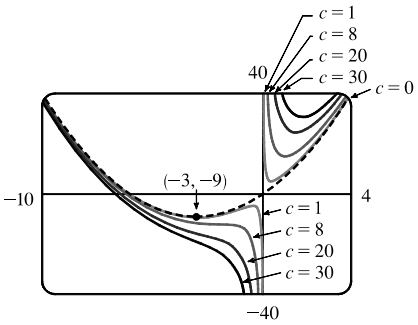
$f''(x) = \dfrac{2(x^3 + c)}{x^3}$, we see that f has an inflection point at $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$. This inflection point moves down and left, approaching the origin as $c \to 0^-$.

f is CU on $(-\infty, 0)$, CD on $\left(0, \sqrt[3]{-c}\right)$, and CU on $\left(\sqrt[3]{-c}, \infty\right)$.

$\boldsymbol{c > 0}$**:** The inflection point $\left(\sqrt[3]{-c}, 6\sqrt[3]{-c}\right)$ is now in the third quadrant and moves up and right, approaching the origin as $c \to 0^+$. f is CU on $\left(-\infty, \sqrt[3]{-c}\right)$, CD on $\left(\sqrt[3]{-c}, 0\right)$, and CU on $(0, \infty)$. f has a local minimum point in the first quadrant. It moves down and left, approaching the origin as $c \to 0^+$. $f'(x) = 0 \quad \Leftrightarrow \quad 2x^3 + 6x^2 - c = h(x) = 0$. Now $h'(x) = 0 \quad \Leftrightarrow \quad x = -2$ or 0, and h (*not* f) has a local maximum at $x = -2$. $h(-2) = 8 - c$, so $c = 8$ makes $h(x) = 0$, and hence, $f'(x) = 0$. When $c > 8$, $f'(x) < 0$ and f is decreasing on $(-\infty, 0)$. For $0 < c < 8$, there is a local minimum that moves toward $(-3, -9)$ and a local maximum that moves toward the origin as c decreases.



$\boldsymbol{c \le 0}$



$\boldsymbol{c \ge 0}$

30. With $c = 0$ in $y = f(x) = x\sqrt{c^2 - x^2}$, the graph of f is just the point $(0, 0)$. Since $(-c)^2 = c^2$, we only consider $c > 0$. Since $f(-x) = -f(x)$, the graph is symmetric about the origin. The domain of f is found by solving $c^2 - x^2 \ge 0 \quad \Leftrightarrow \quad x^2 \le c^2 \quad \Leftrightarrow \quad |x| \le c$, which gives us $[-c, c]$.

$$f'(x) = x \cdot \tfrac{1}{2}(c^2 - x^2)^{-1/2}(-2x) + (c^2 - x^2)^{1/2}(1) = (c^2 - x^2)^{-1/2}[-x^2 + (c^2 - x^2)] = \frac{c^2 - 2x^2}{\sqrt{c^2 - x^2}}.$$

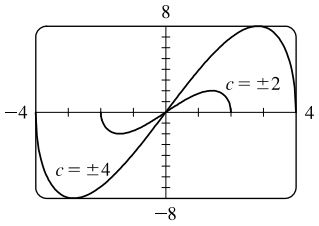
$f'(x) > 0 \quad \Leftrightarrow \quad c^2 - 2x^2 > 0 \quad \Leftrightarrow \quad x^2 < c^2/2 \quad \Leftrightarrow \quad |x| < c/\sqrt{2}$, so f is increasing on $\left(-c/\sqrt{2}, c/\sqrt{2}\right)$ and decreasing on $\left(-c, -c/\sqrt{2}\right)$ and $\left(c/\sqrt{2}, c\right)$. There is a local minimum value of $f\left(-c/\sqrt{2}\right) = \left(-c/\sqrt{2}\right)\sqrt{c^2 - c^2/2} = \left(-c/\sqrt{2}\right)\left(c/\sqrt{2}\right) = -c^2/2$ and a local maximum value of $f\left(c/\sqrt{2}\right) = c^2/2$.

$$f''(x) = \frac{(c^2 - x^2)^{1/2}(-4x) - (c^2 - 2x^2)\frac{1}{2}\left(c^2 - x^2\right)^{-1/2}(-2x)}{[(c^2 - x^2)^{1/2}]^2}$$

$$= \frac{x(c^2 - x^2)^{-1/2}[(c^2 - x^2)(-4) + (c^2 - 2x^2)]}{(c^2 - x^2)^1} = \frac{2x\left(2x^2 - 3c^2\right)}{(c^2 - x^2)^{3/2}},$$

so $f''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \pm\sqrt{\frac{3}{2}}\,c$, but only 0 is in the domain of f.

$f''(x) < 0$ for $0 < x < c$ and $f''(x) > 0$ for $-c < x < 0$, so f is CD on $(0, c)$ and CU on $(-c, 0)$. There is an IP at $(0, 0)$. So as $|c|$ gets larger, the maximum and minimum values increase in magnitude. The value of c does not affect the concavity of f.



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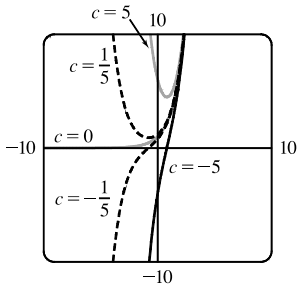
31. $f(x) = e^x + ce^{-x}$. $f = 0 \;\Rightarrow\; ce^{-x} = -e^x \;\Rightarrow\; c = -e^{2x} \;\Rightarrow\; 2x = \ln(-c) \;\Rightarrow\; x = \frac{1}{2}\ln(-c)$.

$f'(x) = e^x - ce^{-x}$. $f' = 0 \;\Rightarrow\; ce^{-x} = e^x \;\Rightarrow\; c = e^{2x} \;\Rightarrow\; 2x = \ln c \;\Rightarrow\; x = \frac{1}{2}\ln c$.

$f''(x) = e^x + ce^{-x} = f(x)$.

The only transitional value of c is 0. As c increases from $-\infty$ to 0, $\frac{1}{2}\ln(-c)$ is both the the x-intercept and inflection point, and this decreases from ∞ to $-\infty$. Also $f' > 0$, so f is increasing. When $c = 0$, $f(x) = f'(x) = f''(x) = e^x$, f is positive, increasing, and concave upward. As c increases from 0 to ∞, the absolute minimum occurs at $x = \frac{1}{2}\ln c$, which increases from $-\infty$ to ∞. Also, $f = f'' > 0$, so f is positive and concave upward. The value of the y-intercept is $f(0) = 1 + c$, and this increases as c increases from $-\infty$ to ∞.

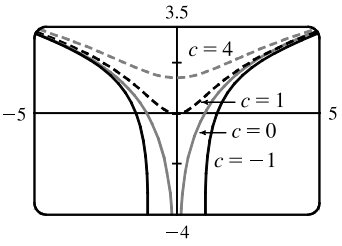


Note: The minimum point $\left(\frac{1}{2}\ln c, 2\sqrt{c}\right)$ can be parameterized by $x = \frac{1}{2}\ln c$, $y = 2\sqrt{c}$, and after eliminating the parameter c, we see that the minimum point lies on the graph of $y = 2e^x$.

32. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and $\lim_{x \to \sqrt{-c}^+} f(x) = \lim_{x \to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim_{x \to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there are no asymptotes. To find the extrema and inflection points, we differentiate:

$f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get $f''(x) = \dfrac{1}{x^2 + c}(2) + 2x\left[-(x^2 + c)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2 + c)^2}$.

Now if $c \le 0$, f'' is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow\; x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $x = \pm\sqrt{c}$, and as c increases, the inflection points get further apart.



33. Note that $c = 0$ is a transitional value at which the graph consists of the x-axis. Also, we can see that if we substitute $-c$ for c, the function $f(x) = \dfrac{cx}{1 + c^2x^2}$ will be reflected in the x-axis, so we investigate only positive values of c (except $c = -1$, as a demonstration of this reflective property). Also, f is an odd function. $\lim_{x \to \pm\infty} f(x) = 0$, so $y = 0$ is a horizontal asymptote for all c. We calculate $f'(x) = \dfrac{(1 + c^2x^2)c - cx(2c^2x)}{(1 + c^2x^2)^2} = -\dfrac{c(c^2x^2 - 1)}{(1 + c^2x^2)^2}$. $f'(x) = 0 \;\Leftrightarrow\; c^2x^2 - 1 = 0 \;\Leftrightarrow$

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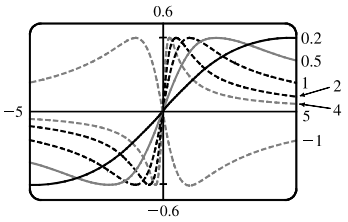
$x = \pm 1/c$. So there is an absolute maximum value of $f(1/c) = \frac{1}{2}$ and an absolute minimum value of $f(-1/c) = -\frac{1}{2}$. These extrema have the same value regardless of c, but the maximum points move closer to the y-axis as c increases.

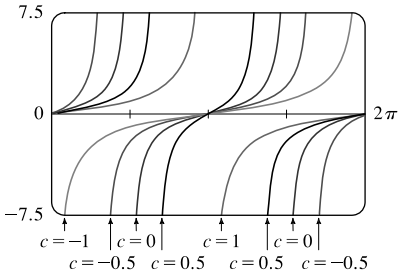
$$f''(x) = \frac{(-2c^3x)(1+c^2x^2)^2 - (-c^3x^2+c)[2(1+c^2x^2)(2c^2x)]}{(1+c^2x^2)^4}$$
$$= \frac{(-2c^3x)(1+c^2x^2) + (c^3x^2-c)(4c^2x)}{(1+c^2x^2)^3} = \frac{2c^3x(c^2x^2-3)}{(1+c^2x^2)^3}$$



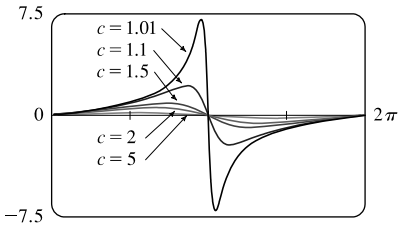
$f''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\pm\sqrt{3}/c$, so there are inflection points at $(0, 0)$ and at $\left(\pm\sqrt{3}/c, \pm\sqrt{3}/4\right)$. Again, the y-coordinate of the inflection points does not depend on c, but as c increases, both inflection points approach the y-axis.

34. $f(x) = \dfrac{\sin x}{c + \cos x} \quad \Rightarrow \quad f'(x) = \dfrac{1 + c\cos x}{\cos^2 x + 2c\cos x + c^2} \quad \Rightarrow \quad f''(x) = \dfrac{\sin x(c\cos x - c^2 + 2)}{\cos^3 x + 3c\cos^2 x + 3c^2\cos x + c^3}$. Notice that f is an odd function and has period 2π. We will graph f for $0 \le x \le 2\pi$.

$|c| \le 1$: See the first figure. f has VAs when the denominator is zero, that is, at $x = \cos^{-1}(-c)$ and $x = 2\pi - \cos^{-1}(-c)$. So for $c = -1$, there are VAs at $x = 0$ and $x = 2\pi$, and as c increases, they move closer to $x = \pi$, which is the single VA when $c = 1$. Note that if $c = 0$, then $f(x) = \tan x$. There are no extreme points (on the entire domain) and inflection points occur at multiples of π.



$c > 1$: See the second figure. $f'(x) = 0 \quad \Leftrightarrow \quad x = \cos^{-1}\left(\dfrac{-1}{c}\right)$ or $x = 2\pi - \cos^{-1}\left(\dfrac{-1}{c}\right)$. The VA disappears and there is now a local maximum and a local minimum. As $c \to 1^+$, the coordinates of the local maximum approach π and ∞, and the coordinates of the local minimum approach π and $-\infty$.



As $c \to \infty$, the graph of f looks like a graph of $y = \sin x$ that is vertically compressed, and the local maximum and local minimum approach $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, respectively.

$f''(x) = 0 \quad \Leftrightarrow \quad \sin x = 0$ (IPs at $x = n\pi$) or $c\cos x - c^2 + 2 = 0$. The second condition is true if $\cos x = \dfrac{c^2-2}{c}$ $[c \ne 0]$. The last equation has two solutions if $-1 < \dfrac{c^2-2}{c} < 1 \quad \Rightarrow \quad -c < c^2 - 2 < c \quad \Rightarrow \quad -c < c^2 - 2$ and $c^2 - 2 < c \quad \Rightarrow \quad c^2 + c - 2 > 0$ and $c^2 - c - 2 < 0 \quad \Rightarrow \quad (c+2)(c-1) > 0$ and $(c-2)(c+1) < 0 \quad \Rightarrow$

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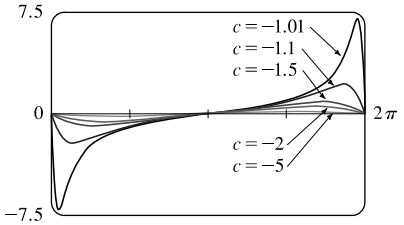
$c-1>0$ [since $c>1$] and $c-2<0 \;\Rightarrow\; c>1$ and $c<2$. Thus, for $1<c<2$, we have 2 nontrivial IPs at $x=\cos^{-1}\left(\dfrac{c^2-2}{c}\right)$ and $x=2\pi-\cos^{-1}\left(\dfrac{c^2-2}{c}\right)$.

c <-1: See the third figure. The VAs for $c=-1$ at $x=0$ and $x=2\pi$ in the first figure disappear and we now have a local minimum and a local maximum. As $c\to -1^+$, the coordinates of the local minimum approach 0 and $-\infty$, and the coordinates of the local maximum approach 2π and ∞. As $c\to-\infty$, the graph of f looks like a graph of $y=\sin x$ that is vertically compressed, and the local minimum and local maximum approach $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$, respectively. As above, we have two nontrivial IPs for $-2<c<-1$.



35. $f(x)=cx+\sin x \;\Rightarrow\; f'(x)=c+\cos x \;\Rightarrow\; f''(x)=-\sin x$

$f(-x)=-f(x)$, so f is an odd function and its graph is symmetric with respect to the origin.

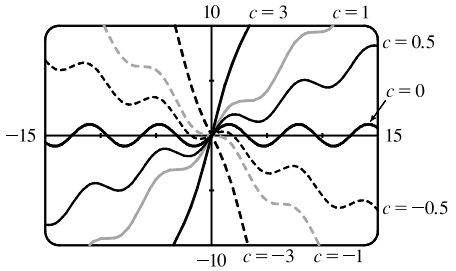
$f(x)=0 \;\Leftrightarrow\; \sin x=-cx$, so 0 is always an x-intercept.

$f'(x)=0 \;\Leftrightarrow\; \cos x=-c$, so there is no critical number when $|c|>1$. If $|c|\le 1$, then there are infinitely many critical numbers. If x_1 is the unique solution of $\cos x=-c$ in the interval $[0,\pi]$, then the critical numbers are $2n\pi\pm x_1$, where n ranges over the integers. (Special cases: When $c=-1$, $x_1=0$; when $c=0$, $x=\frac{\pi}{2}$; and when $c=1$, $x_1=\pi$.)

$f''(x)<0 \;\Leftrightarrow\; \sin x>0$, so f is CD on intervals of the form $(2n\pi,(2n+1)\pi)$. f is CU on intervals of the form $((2n-1)\pi,2n\pi)$. The inflection points of f are the points $(n\pi,n\pi c)$, where n is an integer.

If $c\ge 1$, then $f'(x)\ge 0$ for all x, so f is increasing and has no extremum. If $c\le -1$, then $f'(x)\le 0$ for all x, so f is decreasing and has no extremum. If $|c|<1$, then $f'(x)>0 \;\Leftrightarrow\; \cos x>-c \;\Leftrightarrow\; x$ is in an interval of the form $(2n\pi-x_1,2n\pi+x_1)$ for some integer n. These are the intervals on which f is increasing. Similarly, we find that f is decreasing on the intervals of the form $(2n\pi+x_1,2(n+1)\pi-x_1)$. Thus, f has local maxima at the points $2n\pi+x_1$, where f has the values $c(2n\pi+x_1)+\sin x_1=c(2n\pi+x_1)+\sqrt{1-c^2}$, and f has local minima at the points $2n\pi-x_1$, where we have $f(2n\pi-x_1)=c(2n\pi-x_1)-\sin x_1=c(2n\pi-x_1)-\sqrt{1-c^2}$.

The transitional values of c are -1 and 1. The inflection points move vertically, but not horizontally, when c changes. When $|c|\ge 1$, there is no extremum. For $|c|<1$, the maxima are spaced 2π apart horizontally, as are the minima. The horizontal spacing between maxima and adjacent minima is regular (and equals π) when $c=0$, but the horizontal space between a local maximum and the nearest local minimum shrinks as $|c|$ approaches 1.



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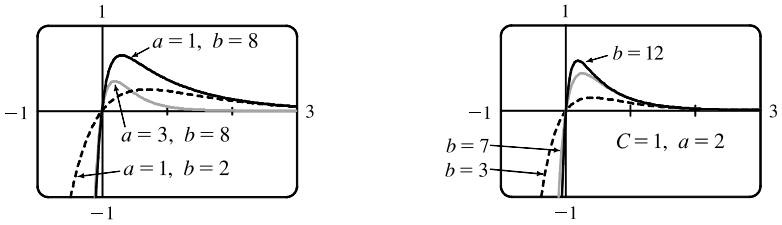
36. For $f(t) = C(e^{-at} - e^{-bt})$, C affects only vertical stretching, so we let $C = 1$. From the first figure, we notice that the graphs all pass through the origin, approach the t-axis as t increases, and approach $-\infty$ as $t \to -\infty$. Next we let $a = 2$ and produce the second figure.



Here, as b increases, the slope of the tangent at the origin increases and the local maximum value increases.

$f(t) = e^{-2t} - e^{-bt} \quad \Rightarrow \quad f'(t) = be^{-bt} - 2e^{-2t}$. $f'(0) = b - 2$, which increases as b increases.

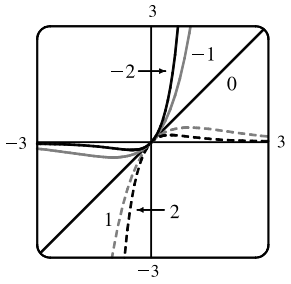
$f'(t) = 0 \quad \Rightarrow \quad be^{-bt} = 2e^{-2t} \quad \Rightarrow \quad \dfrac{b}{2} = e^{(b-2)t} \quad \Rightarrow \quad \ln\dfrac{b}{2} = (b-2)t \quad \Rightarrow \quad t = t_1 = \dfrac{\ln b - \ln 2}{b-2}$, which decreases as b increases (the maximum is getting closer to the y-axis). $f(t_1) = \dfrac{(b-2)2^{2/(b-2)}}{b^{1+2/(b-2)}}$. We can show that this value increases as b increases by considering it to be a function of b and graphing its derivative with respect to b, which is always positive.

37. If $c < 0$, then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} xe^{-cx} = \lim\limits_{x\to-\infty} \dfrac{x}{e^{cx}} \overset{\text{H}}{=} \lim\limits_{x\to-\infty} \dfrac{1}{ce^{cx}} = 0$, and $\lim\limits_{x\to\infty} f(x) = \infty$.

If $c > 0$, then $\lim\limits_{x\to-\infty} f(x) = -\infty$, and $\lim\limits_{x\to\infty} f(x) \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$, respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \quad \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \quad \Leftrightarrow \quad x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \quad \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \quad \Leftrightarrow \quad x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



38. (a) $f(x) = cx^4 - 4x^2 + 1 \quad \Rightarrow \quad f'(x) = 4cx^3 - 8x = 4x(cx^2 - 2)$. If $c \le 0$, then the only real solution of $f'(x) = 0$ is $x = 0$. f' changes from positive to negative at $x = 0$, so f has only a maximum point in this case. If $c > 0$, then $f'(x) = 4x(cx^2 - 2) = 4x\left(\sqrt{c}\,x + \sqrt{2}\right)\left(\sqrt{c}\,x - \sqrt{2}\right)$, and f changes from negative to positive at $x = \pm\sqrt{2}/\sqrt{c}$. Thus, if $c > 0$, the curve has minimum points.

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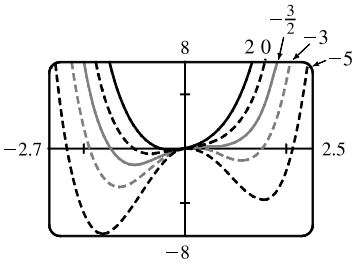
(b) $f\left(\pm\frac{\sqrt{2}}{\sqrt{c}}\right) = c\left(\pm\frac{\sqrt{2}}{c}\right)^4 - 4\left(\pm\frac{\sqrt{2}}{\sqrt{c}}\right)^2 + 1 = \frac{4}{c} - \frac{8}{c} + 1 = -\frac{4}{c} + 1$. For $y = g(x) = -2x^2 + 1$, we have $g\left(\pm\frac{\sqrt{2}}{\sqrt{c}}\right) = -2\left(\pm\frac{\sqrt{2}}{c}\right)^2 + 1 = -\frac{4}{c} + 1$. Also, $f(0) = 1$ and $g(0) = 1$. Thus, the minimum points $\left(\pm\frac{\sqrt{2}}{\sqrt{c}}, -\frac{4}{c} + 1\right)$ and the maximum point $(0,1)$ of every curve in the family $f(x) = cx^4 - 4x^2 + 1$ lie on the parabola $y = -2x^2 + 1$.

39. For $c = 0$, there is no inflection point; the curve is CU everywhere. If c increases, the curve simply becomes steeper, and there are still no inflection points. If c starts at 0 and decreases, a slight upward bulge appears near $x = 0$, so that there are two inflection points for any $c < 0$. This can be seen algebraically by calculating the second derivative: $f(x) = x^4 + cx^2 + x \;\Rightarrow\; f'(x) = 4x^3 + 2cx + 1 \;\Rightarrow\; f''(x) = 12x^2 + 2c$. Thus, $f''(x) > 0$ when $c > 0$. For $c < 0$, there are inflection points when $x = \pm\sqrt{-\frac{1}{6}c}$. For $c = 0$, the graph has one critical number, at the absolute minimum somewhere around $x = -0.6$. As c increases, the number of critical points does not change. If c instead decreases from 0, we see that the graph eventually sprouts another local minimum, to the right of the origin, somewhere between $x = 1$ and $x = 2$. Consequently, there is also a maximum near $x = 0$.

After a bit of experimentation, we find that at $c = -1.5$, there appear to be two critical numbers: the absolute minimum at about $x = -1$, and a horizontal tangent with no extremum at about $x = 0.5$. For any c smaller than this there will be 3 critical points, as shown in the graphs with $c = -3$ and with $c = -5$.



To prove this algebraically, we calculate $f'(x) = 4x^3 + 2cx + 1$. Now if we substitute our value of $c = -1.5$, the formula for $f'(x)$ becomes $4x^3 - 3x + 1 = (x + 1)(2x - 1)^2$. This has a double solution at $x = \frac{1}{2}$, indicating that the function has two critical points: $x = -1$ and $x = \frac{1}{2}$, just as we had guessed from the graph.

40. (a) $f(x) = 2x^3 + cx^2 + 2x \;\Rightarrow\; f'(x) = 6x^2 + 2cx + 2 = 2(3x^2 + cx + 1)$. $f'(x) = 0 \;\Leftrightarrow\; x = \frac{-c \pm \sqrt{c^2 - 12}}{6}$. So f has critical points $\Leftrightarrow\; c^2 - 12 \geq 0 \;\Leftrightarrow\; |c| \geq 2\sqrt{3}$. For $c = \pm 2\sqrt{3}$, $f'(x) \geq 0$ on $(-\infty, \infty)$, so f' does not change signs at $-c/6$, and there is no extremum. If $c^2 - 12 > 0$, then f' changes from positive to negative at $x = \frac{-c - \sqrt{c^2 - 12}}{6}$ and from negative to positive at $x = \frac{-c + \sqrt{c^2 - 12}}{6}$. So f has a local maximum at $x = \frac{-c - \sqrt{c^2 - 12}}{6}$ and a local minimum at $x = \frac{-c + \sqrt{c^2 - 12}}{6}$.

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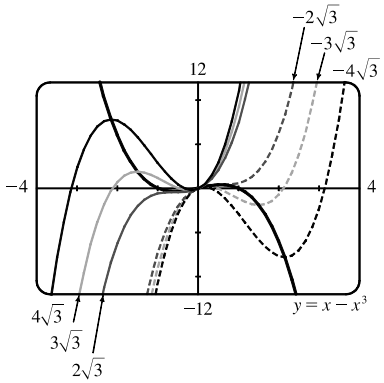
(b) Let x_0 be a critical number for $f(x)$. Then $f'(x_0) = 0 \quad \Rightarrow$

$3x_0^2 + cx_0 + 1 = 0 \quad \Leftrightarrow \quad c = \dfrac{-1 - 3x_0^2}{x_0}$. Now

$$f(x_0) = 2x_0^3 + cx_0^2 + 2x_0 = 2x_0^3 + x_0^2\left(\frac{-1 - 3x_0^2}{x_0}\right) + 2x_0$$

$$= 2x_0^3 - x_0 - 3x_0^3 + 2x_0 = x_0 - x_0^3$$

So the point is $(x_0, y_0) = \left(x_0, x_0 - x_0^3\right)$; that is, the point lies on the curve $y = x - x^3$.



4.7 Optimization Problems

1. (a)

First Number	Second Number	Product
1	22	22
2	21	42
3	20	60
4	19	76
5	18	90
6	17	102
7	16	112
8	15	120
9	14	126
10	13	130
11	12	132

We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

(b) Call the two numbers x and y. Then $x + y = 23$, so $y = 23 - x$. Call the product P. Then $P = xy = x(23 - x) = 23x - x^2$, so we wish to maximize the function $P(x) = 23x - x^2$. Since $P'(x) = 23 - 2x$, we see that $P'(x) = 0 \quad \Leftrightarrow \quad x = \frac{23}{2} = 11.5$. Thus, the maximum value of P is $P(11.5) = (11.5)^2 = 132.25$ and it occurs when $x = y = 11.5$.

Or: Note that $P''(x) = -2 < 0$ for all x, so P is everywhere concave downward and the local maximum at $x = 11.5$ must be an absolute maximum.

2. The two numbers are $x + 100$ and x. Minimize $f(x) = (x + 100)x = x^2 + 100x$. $f'(x) = 2x + 100 = 0 \quad \Rightarrow \quad x = -50$. Since $f''(x) = 2 > 0$, there is an absolute minimum at $x = -50$. The two numbers are 50 and -50.

3. The two numbers are x and $\dfrac{100}{x}$, where $x > 0$. Minimize $f(x) = x + \dfrac{100}{x}$. $f'(x) = 1 - \dfrac{100}{x^2} = \dfrac{x^2 - 100}{x^2}$. The critical number is $x = 10$. Since $f'(x) < 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$, there is an absolute minimum at $x = 10$. The numbers are 10 and 10.

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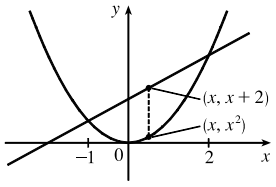
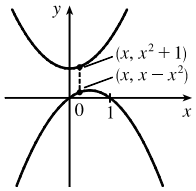
4. Call the two numbers x and y. Then $x+y=16$, so $y=16-x$. Call the sum of their squares S. Then $S=x^2+y^2=x^2+(16-x)^2 \quad \Rightarrow \quad S'=2x+2(16-x)(-1)=2x-32+2x=4x-32$. $S'=0 \quad \Rightarrow \quad x=8$. Since $S'(x)<0$ for $0<x<8$ and $S'(x)>0$ for $x>8$, there is an absolute minimum at $x=8$. Thus, $y=16-8=8$ and $S=8^2+8^2=128$.

5. Let the vertical distance be given by $v(x)=(x+2)-x^2$, $-1\le x\le 2$. $v'(x)=1-2x=0 \quad \Leftrightarrow \quad x=\frac{1}{2}$. $v(-1)=0$, $v\left(\frac{1}{2}\right)=\frac{9}{4}$, and $v(2)=0$, so there is an absolute maximum at $x=\frac{1}{2}$. The maximum distance is $v\left(\frac{1}{2}\right)=\frac{1}{2}+2-\frac{1}{4}=\frac{9}{4}$.



6. Let the vertical distance be given by $v(x)=(x^2+1)-(x-x^2)=2x^2-x+1$. $v'(x)=4x-1=0 \quad \Leftrightarrow \quad x=\frac{1}{4}$. $v'(x)<0$ for $x<\frac{1}{4}$ and $v'(x)>0$ for $x>\frac{1}{4}$, so there is an absolute minimum at $x=\frac{1}{4}$. The minimum distance is $v\left(\frac{1}{4}\right)=\frac{1}{8}-\frac{1}{4}+1=\frac{7}{8}$.



7. If the rectangle has dimensions x and y, then its perimeter is $2x+2y=100$ m, so $y=50-x$. Thus, the area is $A=xy=x(50-x)$. We wish to maximize the function $A(x)=x(50-x)=50x-x^2$, where $0<x<50$. Since $A'(x)=50-2x=-2(x-25)$, $A'(x)>0$ for $0<x<25$ and $A'(x)<0$ for $25<x<50$. Thus, A has an absolute maximum at $x=25$, and $A(25)=25^2=625$ m^2. The dimensions of the rectangle that maximize its area are $x=y=25$ m. (The rectangle is a square.)

8. If the rectangle has dimensions x and y, then its area is $xy=1000$ m^2, so $y=1000/x$. The perimeter $P=2x+2y=2x+2000/x$. We wish to minimize the function $P(x)=2x+2000/x$ for $x>0$. $P'(x)=2-2000/x^2=(2/x^2)(x^2-1000)$, so the only critical number in the domain of P is $x=\sqrt{1000}$. $P''(x)=4000/x^3>0$, so P is concave upward throughout its domain and $P\left(\sqrt{1000}\right)=4\sqrt{1000}$ is an absolute minimum value. The dimensions of the rectangle with minimal perimeter are $x=y=\sqrt{1000}=10\sqrt{10}$ m. (The rectangle is a square.)

9. We need to maximize Y for $N\ge 0$. $Y(N)=\dfrac{kN}{1+N^2} \quad \Rightarrow$

$$Y'(N)=\frac{(1+N^2)k-kN(2N)}{(1+N^2)^2}=\frac{k(1-N^2)}{(1+N^2)^2}=\frac{k(1+N)(1-N)}{(1+N^2)^2}.$$

$Y'(N)>0$ for $0<N<1$ and $Y'(N)<0$ for $N>1$. Thus, Y has an absolute maximum of $Y(1)=\frac{1}{2}k$ at $N=1$.

10. We need to maximize P for $I\ge 0$. $P(I)=\dfrac{100I}{I^2+I+4} \quad \Rightarrow$

$$P'(I)=\frac{(I^2+I+4)(100)-100I(2I+1)}{(I^2+I+4)^2}=\frac{100(I^2+I+4-2I^2-I)}{(I^2+I+4)^2}=\frac{-100(I^2-4)}{(I^2+I+4)^2}=\frac{-100(I+2)(I-2)}{(I^2+I+4)^2}.$$

$P'(I)>0$ for $0<I<2$ and $P'(I)<0$ for $I>2$. Thus, P has an absolute maximum of $P(2)=20$ at $I=2$.

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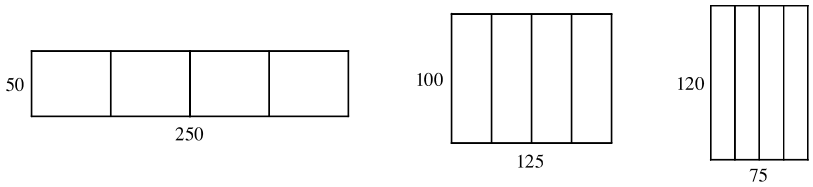
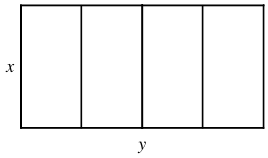
11. (a)



The areas of the three figures are 12,500, 12,500, and 9000 ft^2. There appears to be a maximum area of at least 12,500 ft^2.

(b) Let x denote the length of each of two sides and three dividers.

Let y denote the length of the other two sides.



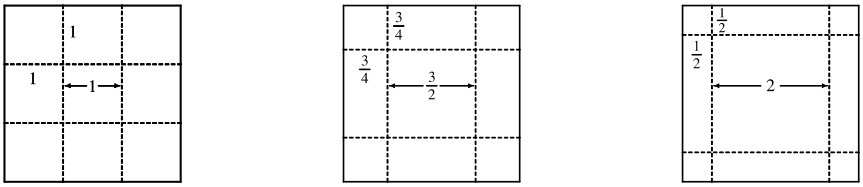
(c) Area $A =$ length $\times$ width $= y \cdot x$

(d) Length of fencing $= 750 \quad \Rightarrow \quad 5x + 2y = 750$

(e) $5x + 2y = 750 \quad \Rightarrow \quad y = 375 - \frac{5}{2}x \quad \Rightarrow \quad A(x) = \left(375 - \frac{5}{2}x\right)x = 375x - \frac{5}{2}x^2$

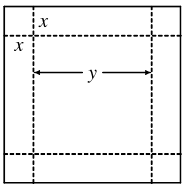
(f) $A'(x) = 375 - 5x = 0 \quad \Rightarrow \quad x = 75$. Since $A''(x) = -5 < 0$ there is an absolute maximum when $x = 75$. Then $y = \frac{375}{2} = 187.5$. The largest area is $75\left(\frac{375}{2}\right) = 14{,}062.5$ ft^2. These values of x and y are between the values in the first and second figures in part (a). Our original estimate was low.

12. (a)



The volumes of the resulting boxes are 1, 1.6875, and 2 ft^3. There appears to be a maximum volume of at least 2 ft^3.

(b) Let x denote the length of the side of the square being cut out. Let y denote the length of the base.



(c) Volume $V =$ length $\times$ width $\times$ height $\quad \Rightarrow \quad V = y \cdot y \cdot x = xy^2$

(d) Length of cardboard $= 3 \quad \Rightarrow \quad x + y + x = 3 \quad \Rightarrow \quad y + 2x = 3$

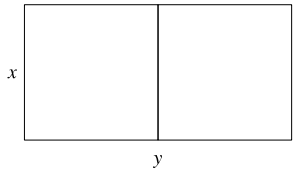
(e) $y + 2x = 3 \quad \Rightarrow \quad y = 3 - 2x \quad \Rightarrow \quad V(x) = x(3 - 2x)^2$

(f) $V(x) = x(3 - 2x)^2 \quad \Rightarrow$

$V'(x) = x \cdot 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3)$,

so the critical numbers are $x = \frac{3}{2}$ and $x = \frac{1}{2}$. Now $0 \le x \le \frac{3}{2}$ and $V(0) = V\left(\frac{3}{2}\right) = 0$, so the maximum is $V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)(2)^2 = 2$ ft^3, which is the value found from our third figure in part (a).

13.



$xy = 1.5 \times 10^6$, so $y = 1.5 \times 10^6/x$. Minimize the amount of fencing, which is $3x + 2y = 3x + 2(1.5 \times 10^6/x) = 3x + 3 \times 10^6/x = F(x)$.

$F'(x) = 3 - 3 \times 10^6/x^2 = 3(x^2 - 10^6)/x^2$. The critical number is $x = 10^3$ and $F'(x) < 0$ for $0 < x < 10^3$ and $F'(x) > 0$ if $x > 10^3$, so the absolute minimum occurs when $x = 10^3$ and $y = 1.5 \times 10^3$.

The field should be 1000 feet by 1500 feet with the middle fence parallel to the short side of the field.

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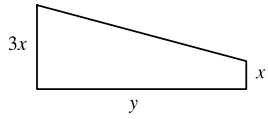
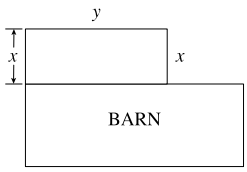
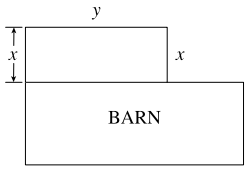
14. 

See the figure. We have $4x + y = 1200 \quad \Rightarrow \quad y = 1200 - 4x$. The trapezoidal area A is $\frac{1}{2}(x + 3x)y = 2xy = 2x(1200 - 4x) = 2400x - 8x^2,\ 0 < x < 300$. $A'(x) = 2400 - 16x = 0 \quad \Leftrightarrow \quad x = 150$. Since $A'(x) > 0$ for $0 < x < 150$ and $A'(x) < 0$ for $150 < x < 300$, there is an absolute maximum when $x = 150$ by the First Derivative Test for Absolute Extreme Values. $x = 150 \quad \Rightarrow \quad y = 1200 - 4(150) = 600$. The maximum area is $2xy = 2(150)(600) = 180{,}000 \text{ ft}^2$.

15. 

See the figure. The fencing cost \$20 per linear foot to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(20x) + 20y + 20x = 20y + 30x$. The area A will be maximized when $C = 5000$, so $5000 = 20y + 30x \quad \Leftrightarrow \quad 20y = 5000 - 30x \quad \Leftrightarrow$ $y = 250 - \frac{3}{2}x$. Now $A = xy = x\left(250 - \frac{3}{2}x\right) = 250x - \frac{3}{2}x^2 \quad \Rightarrow \quad A' = 250 - 3x. \quad A' = 0 \quad \Leftrightarrow \quad x = \frac{250}{3}$ and since $A'' = -3 < 0$, we have a maximum for A when $x = \frac{250}{3}$ ft and $y = 250 - \frac{3}{2}\left(\frac{250}{3}\right) = 125$ ft. [The maximum area is $125\left(\frac{250}{3}\right) = 10{,}416.\overline{6} \text{ ft}^2$.]

16. 

See the figure. The fencing cost \$20 per linear foot to install and the cost of the fencing on the west side will be split with the neighbor, so the farmer's cost C will be $C = \frac{1}{2}(20x) + 20y + 20x = 20y + 30x$. The area A to be enclosed is 8000 ft^2, so $A = xy = 8000 \quad \Rightarrow \quad y = \dfrac{8000}{x}$.

Now $C = 20y + 30x = 20\left(\dfrac{8000}{x}\right) + 30x = \dfrac{160{,}000}{x} + 30x \quad \Rightarrow \quad C' = -\dfrac{160{,}000}{x^2} + 30. \quad C' = 0 \quad \Leftrightarrow$ $30 = \dfrac{160{,}000}{x^2} \quad \Leftrightarrow \quad x^2 = \dfrac{16{,}000}{3} \quad \Rightarrow \quad x = \sqrt{\dfrac{16{,}000}{3}} = 40\sqrt{\dfrac{10}{3}} = \dfrac{40}{3}\sqrt{30}$. Since $C'' = \dfrac{320{,}000}{x^3} > 0$ [for $x > 0$], we have a minimum for C when $x = \dfrac{40}{3}\sqrt{30}$ ft and $y = \dfrac{8000}{x} = \dfrac{8000}{40} \cdot \dfrac{3}{\sqrt{30}} \cdot \dfrac{\sqrt{30}}{\sqrt{30}} = 20\sqrt{30}$ ft. [The minimum cost is $20(20\sqrt{30}) + 30\left(\frac{40}{3}\sqrt{30}\right) = 800\sqrt{30} \approx \$4381.78$.]

17. (a) Let the rectangle have sides x and y and area A, so $A = xy$ or $y = A/x$. The problem is to minimize the perimeter $= 2x + 2y = 2x + 2A/x = P(x)$. Now $P'(x) = 2 - 2A/x^2 = 2\left(x^2 - A\right)/x^2$. So the critical number is $x = \sqrt{A}$. Since $P'(x) < 0$ for $0 < x < \sqrt{A}$ and $P'(x) > 0$ for $x > \sqrt{A}$, there is an absolute minimum at $x = \sqrt{A}$. The sides of the rectangle are $\sqrt{A}$ and $A/\sqrt{A} = \sqrt{A}$, so the rectangle is a square.

(b) Let p be the perimeter and x and y the lengths of the sides, so $p = 2x + 2y \quad \Rightarrow \quad 2y = p - 2x \quad \Rightarrow \quad y = \frac{1}{2}p - x$. The area is $A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2$. Now $A'(x) = 0 \quad \Rightarrow \quad \frac{1}{2}p - 2x = 0 \quad \Rightarrow \quad 2x = \frac{1}{2}p \quad \Rightarrow \quad x = \frac{1}{4}p$. Since $A''(x) = -2 < 0$, there is an absolute maximum for A when $x = \frac{1}{4}p$ by the Second Derivative Test. The sides of the rectangle are $\frac{1}{4}p$ and $\frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p$, so the rectangle is a square.

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18. Let b be the length of the base of the box and h the height. The volume is $32{,}000 = b^2h \quad \Rightarrow \quad h = 32{,}000/b^2$.

The surface area of the open box is $S = b^2 + 4hb = b^2 + 4(32{,}000/b^2)b = b^2 + 4(32{,}000)/b$.

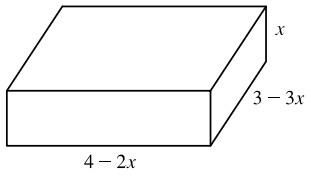
So $S'(b) = 2b - 4(32{,}000)/b^2 = 2\big(b^3 - 64{,}000\big)/b^2 = 0 \quad \Leftrightarrow \quad b = \sqrt[3]{64{,}000} = 40$. This gives an absolute minimum since $S'(b) < 0$ if $0 < b < 40$ and $S'(b) > 0$ if $b > 40$. The box should be $40 \times 40 \times 20$.

19. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \quad \Rightarrow \quad h = (1200 - b^2)/(4b)$.

The volume is $V = b^2h = b^2(1200 - b^2)/4b = 300b - b^3/4 \quad \Rightarrow \quad V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \quad \Rightarrow \quad 300 = \frac{3}{4}b^2 \quad \Rightarrow \quad b^2 = 400 \quad \Rightarrow \quad b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values. If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2h = (20)^2(10) = 4000 \text{ cm}^3$.

20.



$V = x(4 - 2x)(3 - 3x) = 12x - 18x^2 + 6x^3, \ 0 \le x \le 1$.

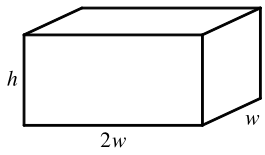
$V'(x) = 12 - 36x + 18x^2 = 6(2 - 6x + 3x^2)$.

$V'(x) = 0 \quad \Rightarrow \quad x = \dfrac{3 - \sqrt{3}}{3}$ by the quadratic formula (rejecting $x = \dfrac{3 + \sqrt{3}}{3}$ since it is greater than 1). $V''(x) = -36 + 36x \quad \Rightarrow \quad V''\left(\dfrac{3 - \sqrt{3}}{3}\right) = -36 + 36\left(\dfrac{3 - \sqrt{3}}{3}\right) = -36 + 36 - 12\sqrt{3} < 0$,

so there is an absolute maximum at $x = \dfrac{3 - \sqrt{3}}{3}$ since $V(0) = V(1) = 0$. The maximum volume is

$V\left(\dfrac{3 - \sqrt{3}}{3}\right) = \dfrac{4\sqrt{3}}{3} \approx 2.31 \text{ ft}^3$.

21.



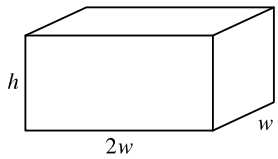
$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$.

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] = 20w^2 + 36wh$, so

$C(w) = 20w^2 + 36w\big(5/w^2\big) = 20w^2 + 180/w$.

$C'(w) = 40w - 180/w^2 = (40w^3 - 180)/w^2 = 40\left(w^3 - \frac{9}{2}\right)\big/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{9}{2}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{9}{2}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{9}{2}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{9}{2}}$. The minimum cost is $C\Big(\sqrt[3]{\frac{9}{2}}\Big) = 20\Big(\sqrt[3]{\frac{9}{2}}\Big)^2 + \dfrac{180}{\sqrt[3]{9/2}} \approx \$163.54$.

22.



$V = lwh \quad \Rightarrow \quad 10 = (2w)(w)h = 2w^2h$, so $h = 5/w^2$.

The cost is $10(2w^2) + 6[2(2wh) + 2(hw)] + 6(2w^2) = 32w^2 + 36wh$, so

$C(w) = 32w^2 + 36w(5/w^2) = 32w^2 + 180/w$.

$C'(w) = 64w - 180/w^2 = (64w^3 - 180)/w^2 = 4(16w^3 - 45)/w^2 \quad \Rightarrow \quad w = \sqrt[3]{\frac{45}{16}}$ is the critical number. There is an absolute minimum for C when $w = \sqrt[3]{\frac{45}{16}}$ since $C'(w) < 0$ for $0 < w < \sqrt[3]{\frac{45}{16}}$ and $C'(w) > 0$ for $w > \sqrt[3]{\frac{45}{16}}$. The minimum cost is $C\Big(\sqrt[3]{\frac{45}{16}}\Big) = 32\Big(\sqrt[3]{\frac{45}{16}}\Big)^2 + \dfrac{180}{\sqrt[3]{45/16}} \approx \$191.28$.

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23. Let $x > 0$ be the length of the package and $y > 0$ be the length of the sides of the square base. We have $x + 4y = 108 \Rightarrow x = 108 - 4y$. The volume is $V = xy^2 = (108 - 4y)y^2 = 108y^2 - 4y^3$ [for $0 < y < 27$].

$V'(y) = 216y - 12y^2 = 12y(18 - y) = 0 \Rightarrow y = 18$ [since $y \neq 0$]. Since $V'(y) > 0$ for $0 < y < 18$ and $V'(y) < 0$ for $y > 18$, there is an absolute maximum when $y = 18$ by the First Derivative Test for Absolute Extreme Values. If $y = 18$, then $x = 108 - 4(18) = 36$, so the dimensions that give the greatest volume are 18 in $\times$ 18 in $\times$ 36 in, giving a greatest possible volume of 11,664 in^3.

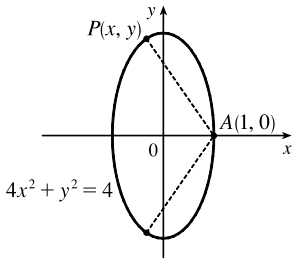
24. Let $x > 0$ be the length of the package and $r > 0$ be the radius of the circular base. We have $x + 2\pi r = 108 \Rightarrow x = 108 - 2\pi r$. The volume is $V = \pi r^2 x = \pi r^2(108 - 2\pi r) = \pi(108r^2 - 2\pi r^3)$ [for $0 < r < 54/\pi$].

$V'(r) = \pi(216r - 6\pi r^2) = 6\pi r(36 - \pi r) = 0 \Rightarrow r = \dfrac{36}{\pi}$ [since $r \neq 0$]. Since $V'(r) > 0$ for $0 < r < \dfrac{36}{\pi}$ and $V'(r) < 0$ for $r > \dfrac{36}{\pi}$, there is an absolute maximum when $r = \dfrac{36}{\pi}$ by the First Derivative Test for Absolute Extreme Values. If $r = \dfrac{36}{\pi}$, then $x = 108 - 2\pi\left(\dfrac{36}{\pi}\right) = 108 - 72 = 36$, so the dimensions that give the greatest volume are a length of 36 in. and a base radius of $\dfrac{36}{\pi} \approx 11.46$ in., giving a greatest possible volume of $\dfrac{46{,}656}{\pi}$ in^3 [$\approx 14{,}851.1$ in^3].

25. The distance d from the origin $(0, 0)$ to a point $(x, 2x + 3)$ on the line is given by $d = \sqrt{(x - 0)^2 + (2x + 3 - 0)^2}$ and the square of the distance is $S = d^2 = x^2 + (2x + 3)^2$. $S' = 2x + 2(2x + 3)2 = 10x + 12$ and $S' = 0 \Leftrightarrow x = -\frac{6}{5}$. Now $S'' = 10 > 0$, so we know that S has a minimum at $x = -\frac{6}{5}$. Thus, the y-value is $2\left(-\frac{6}{5}\right) + 3 = \frac{3}{5}$ and the point is $\left(-\frac{6}{5}, \frac{3}{5}\right)$.

26. The distance d from the point $(3, 0)$ to a point $(x, \sqrt{x}\,)$ on the curve is given by $d = \sqrt{(x - 3)^2 + \left(\sqrt{x} - 0\right)^2}$ and the square of the distance is $S = d^2 = (x - 3)^2 + x$. $S' = 2(x - 3) + 1 = 2x - 5$ and $S' = 0 \Leftrightarrow x = \frac{5}{2}$. Now $S'' = 2 > 0$, so we know that S has a minimum at $x = \frac{5}{2}$. Thus, the y-value is $\sqrt{\frac{5}{2}}$ and the point is $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$.

27.



From the figure, we see that there are two points that are farthest away from $A(1, 0)$. The distance d from A to an arbitrary point $P(x, y)$ on the ellipse is $d = \sqrt{(x - 1)^2 + (y - 0)^2}$ and the square of the distance is $S = d^2 = x^2 - 2x + 1 + y^2 = x^2 - 2x + 1 + (4 - 4x^2) = -3x^2 - 2x + 5$. $S' = -6x - 2$ and $S' = 0 \Rightarrow x = -\frac{1}{3}$. Now $S'' = -6 < 0$, so we know that S has a maximum at $x = -\frac{1}{3}$. Since $-1 \le x \le 1$, $S(-1) = 4$, $S\left(-\frac{1}{3}\right) = \frac{16}{3}$, and $S(1) = 0$, we see that the maximum distance is $\sqrt{\frac{16}{3}}$. The corresponding y-values are $y = \pm\sqrt{4 - 4\left(-\frac{1}{3}\right)^2} = \pm\sqrt{\frac{32}{9}} = \pm\frac{4}{3}\sqrt{2} \approx \pm 1.89$. The points are $\left(-\frac{1}{3}, \pm\frac{4}{3}\sqrt{2}\right)$.

28. The distance d from the point $(4, 2)$ to a point $(x, \sin x)$ on the curve is given by $d = \sqrt{(x - 4)^2 + (\sin x - 2)^2}$ and the square of the distance is $S = d^2 = (x - 4)^2 + (\sin x - 2)^2$. $S' = 2(x - 4) + 2(\sin x - 2)\cos x$. Using a calculator, it is

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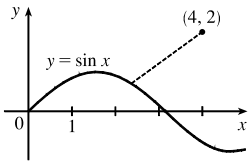
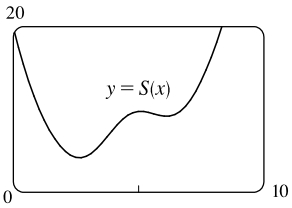
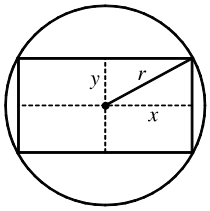
clear that S has a minimum between 0 and 5, and from a graph of S', we find that $S' = 0 \quad \Rightarrow \quad x \approx 2.65$, so the point is about $(2.65, 0.47)$.





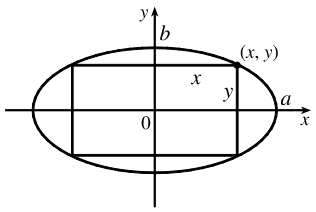
29.



The area of the rectangle is $(2x)(2y) = 4xy$. Also $r^2 = x^2 + y^2$ so $y = \sqrt{r^2 - x^2}$, so the area is $A(x) = 4x\sqrt{r^2 - x^2}$. Now $A'(x) = 4\left(\sqrt{r^2 - x^2} - \dfrac{x^2}{\sqrt{r^2 - x^2}}\right) = 4\,\dfrac{r^2 - 2x^2}{\sqrt{r^2 - x^2}}$. The critical number is $x = \frac{1}{\sqrt{2}}r$. Clearly this gives a maximum.

$y = \sqrt{r^2 - \left(\frac{1}{\sqrt{2}}r\right)^2} = \sqrt{\frac{1}{2}r^2} = \frac{1}{\sqrt{2}}r = x$, which tells us that the rectangle is a square. The dimensions are $2x = \sqrt{2}\,r$ and $2y = \sqrt{2}\,r$.

30.

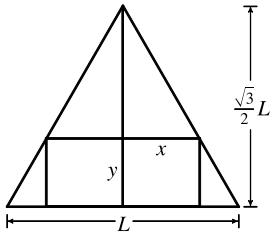


The area of the rectangle is $(2x)(2y) = 4xy$. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ gives $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, so we maximize $A(x) = 4\dfrac{b}{a}x\sqrt{a^2 - x^2}$.

$$A'(x) = \frac{4b}{a}\left[x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2} \cdot 1\right]$$
$$= \frac{4b}{a}\,(a^2 - x^{2-1/2}[-x^2 + a^2 - x^2] = \frac{4b}{a\sqrt{a^2 - x^2}}[a^2 - 2x^2]$$

So the critical number is $x = \frac{1}{\sqrt{2}}\,a$, and this clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}\,b$, so the maximum area is $4\left(\frac{1}{\sqrt{2}}\,a\right)\left(\frac{1}{\sqrt{2}}\,b\right) = 2ab$.

31.



The height h of the equilateral triangle with sides of length L is $\frac{\sqrt{3}}{2}\,L$, since $h^2 + (L/2)^2 = L^2 \quad \Rightarrow \quad h^2 = L^2 - \frac{1}{4}L^2 = \frac{3}{4}L^2 \quad \Rightarrow$ $h = \frac{\sqrt{3}}{2}L$. Using similar triangles, $\dfrac{\frac{\sqrt{3}}{2}L - y}{x} = \dfrac{\frac{\sqrt{3}}{2}L}{L/2} = \sqrt{3} \quad \Rightarrow$ $\sqrt{3}\,x = \frac{\sqrt{3}}{2}L - y \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}L - \sqrt{3}\,x \quad \Rightarrow \quad y = \frac{\sqrt{3}}{2}(L - 2x)$.

The area of the inscribed rectangle is $A(x) = (2x)y = \sqrt{3}\,x(L - 2x) = \sqrt{3}\,Lx - 2\sqrt{3}\,x^2$, where $0 \le x \le L/2$. Now $0 = A'(x) = \sqrt{3}\,L - 4\sqrt{3}\,x \quad \Rightarrow \quad x = \sqrt{3}\,L/\left(4\sqrt{3}\right) = L/4$. Since $A(0) = A(L/2) = 0$, the maximum occurs when $x = L/4$, and $y = \frac{\sqrt{3}}{2}L - \frac{\sqrt{3}}{4}L = \frac{\sqrt{3}}{4}L$, so the dimensions are $L/2$ and $\frac{\sqrt{3}}{4}L$.

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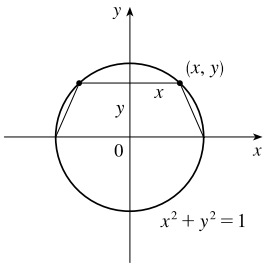
32.

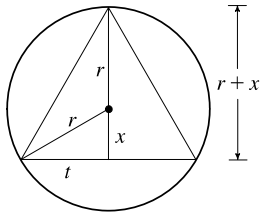


The area A of a trapezoid is given by $A = \frac{1}{2}h(B + b)$. From the diagram, $h = y$, $B = 2$, and $b = 2x$, so $A = \frac{1}{2}y(2 + 2x) = y(1 + x)$. Since it's easier to substitute for y^2, we'll let $T = A^2 = y^2(1 + x)^2 = (1 - x^2)(1 + x)^2$. Now

$$T' = (1 - x^2)2(1 + x) + (1 + x)^2(-2x) = -2(1 + x)[-(1 - x^2) + (1 + x)x]$$
$$= -2(1 + x)(2x^2 + x - 1) = -2(1 + x)(2x - 1)(x + 1)$$

$T' = 0 \quad \Leftrightarrow \quad x = -1$ or $x = \frac{1}{2}$. $T' > 0$ if $x < \frac{1}{2}$ and $T' < 0$ if $x > \frac{1}{2}$, so we get a maximum at $x = \frac{1}{2}$ [$x = -1$ gives us $A = 0$]. Thus, $y = \sqrt{1 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}$ and the maximum area is $A = y(1 + x) = \frac{\sqrt{3}}{2}\left(1 + \frac{1}{2}\right) = \frac{3\sqrt{3}}{4}$.

33.



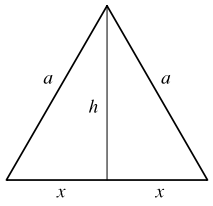
The area of the triangle is $A(x) = \frac{1}{2}(2t)(r + x) = t(r + x) = \sqrt{r^2 - x^2}(r + x)$. Then

$$0 = A'(x) = r\frac{-2x}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} + x\frac{-2x}{2\sqrt{r^2 - x^2}}$$
$$= -\frac{x^2 + rx}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2} \quad \Rightarrow$$

$\frac{x^2 + rx}{\sqrt{r^2 - x^2}} = \sqrt{r^2 - x^2} \quad \Rightarrow \quad x^2 + rx = r^2 - x^2 \quad \Rightarrow \quad 0 = 2x^2 + rx - r^2 = (2x - r)(x + r) \quad \Rightarrow$

$x = \frac{1}{2}r$ or $x = -r$. Now $A(r) = 0 = A(-r) \quad \Rightarrow$ the maximum occurs where $x = \frac{1}{2}r$, so the triangle has height $r + \frac{1}{2}r = \frac{3}{2}r$ and base $2\sqrt{r^2 - \left(\frac{1}{2}r\right)^2} = 2\sqrt{\frac{3}{4}r^2} = \sqrt{3}\,r$.

34.



From the figure, we have $x^2 + h^2 = a^2 \quad \Rightarrow \quad h = \sqrt{a^2 - x^2}$. The area of the isosceles triangle is $A = \frac{1}{2}(2x)h = xh = x\sqrt{a^2 - x^2}$ with $0 \le x \le a$. Now

$$A' = x \cdot \tfrac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2}(1)$$
$$= (a^2 - x^2)^{-1/2}[-x^2 + (a^2 - x^2)] = \frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$$

$A' = 0 \quad \Leftrightarrow \quad x^2 = \frac{1}{2}a^2 \quad \Rightarrow \quad x = a/\sqrt{2}$. Since $A(0) = 0$, $A(a) = 0$, and $A(a/\sqrt{2}) = (a/\sqrt{2})\sqrt{a^2/2} = \frac{1}{2}a^2$, we see that $x = a/\sqrt{2}$ gives us the maximum area and the length of the base is $2x = 2(a/\sqrt{2}) = \sqrt{2}\,a$. Note that the triangle has sides a, a, and $\sqrt{2}\,a$, which form a *right* triangle, with the right angle between the two sides of equal length.

35. The area of the triangle is $A = \frac{1}{2}a(2a)\sin\theta$ for $0 < \theta < \pi$. $A'(\theta) = a^2\cos\theta = 0 \quad \Rightarrow \quad \cos\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}$. Since $A'(\theta) > 0$ for $0 < \theta < \frac{\pi}{2}$ and $A'(\theta) < 0$ for $\frac{\pi}{2} < \theta < \pi$, there is an absolute maximum when $\theta = \frac{\pi}{2}$ by the First Derivative Test for Absolute Extreme Values. (The maximum area of $\frac{1}{2}a(2a)\sin\frac{\pi}{2} = a^2$ results from the triangle being a right triangle.)

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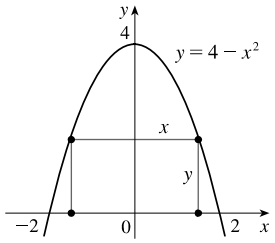
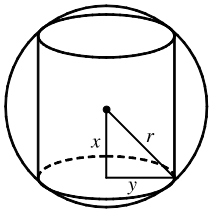
36.



The area of the rectangle is $A = 2xy = 2x(4 - x^2) = 8x - 2x^3,\ 0 < x < 2.$

$A'(x) = 8 - 6x^2 = 2(4 - 3x^2) = 0 \quad \Rightarrow \quad x = \dfrac{2}{\sqrt{3}}$ [since $x > 0$]. Since $A'(x) > 0$ for $0 < x < \dfrac{2}{\sqrt{3}}$ and $A'(x) < 0$ for $\dfrac{2}{\sqrt{3}} < x < 2$, there is an absolute maximum when $x = \dfrac{2}{\sqrt{3}}$ by the First Derivative Test for Absolute Extreme Values. Thus, the largest possible area of the rectangle is $A = 2\left(\dfrac{2}{\sqrt{3}}\right)\left[4 - \left(\dfrac{2}{\sqrt{3}}\right)^2\right] = \left(\dfrac{4}{\sqrt{3}}\right)\left(\dfrac{8}{3}\right) = \dfrac{32}{3\sqrt{3}} \approx 6.16.$

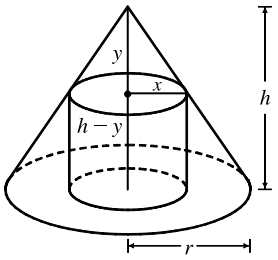
37.



The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2$, so $V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$.

$V'(x) = 2\pi\left(r^2 - 3x^2\right) = 0 \quad \Rightarrow \quad x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V\left(r/\sqrt{3}\,\right) = \pi(r^2 - r^2/3)\left(2r/\sqrt{3}\,\right) = 4\pi r^3/\left(3\sqrt{3}\,\right)$.

38.



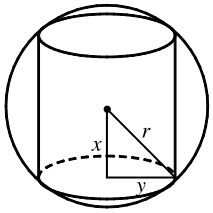
By similar triangles, $y/x = h/r$, so $y = hx/r$. The volume of the cylinder is $\pi x^2(h - y) = \pi hx^2 - (\pi h/r)x^3 = V(x)$. Now

$V'(x) = 2\pi hx - (3\pi h/r)x^2 = \pi hx(2 - 3x/r)$.

So $V'(x) = 0 \quad \Rightarrow \quad x = 0$ or $x = \frac{2}{3}r$. The maximum clearly occurs when $x = \frac{2}{3}r$ and then the volume is

$\pi hx^2 - (\pi h/r)x^3 = \pi hx^2(1 - x/r) = \pi\left(\frac{2}{3}r\right)^2 h\left(1 - \frac{2}{3}\right) = \frac{4}{27}\pi r^2 h.$

39.



The cylinder has surface area

$$\begin{aligned} 2(\text{area of the base}) + (\text{lateral surface area}) &= 2\pi(\text{radius})^2 + 2\pi(\text{radius})(\text{height}) \\ &= 2\pi y^2 + 2\pi y(2x) \end{aligned}$$

Now $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2 \quad \Rightarrow \quad y = \sqrt{r^2 - x^2}$, so the surface area is

$$\begin{aligned} S(x) &= 2\pi(r^2 - x^2) + 4\pi x\sqrt{r^2 - x^2},\ \ 0 \le x \le r \\ &= 2\pi r^2 - 2\pi x^2 + 4\pi\left(x\sqrt{r^2 - x^2}\,\right) \end{aligned}$$

Thus,
$$\begin{aligned} S'(x) &= 0 - 4\pi x + 4\pi\left[x \cdot \tfrac{1}{2}(r^2 - x^2)^{-1/2}(-2x) + (r^2 - x^2)^{1/2} \cdot 1\right] \\ &= 4\pi\left[-x - \frac{x^2}{\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right] = 4\pi \cdot \frac{-x\sqrt{r^2 - x^2} - x^2 + r^2 - x^2}{\sqrt{r^2 - x^2}} \end{aligned}$$

$S'(x) = 0 \quad \Rightarrow \quad x\sqrt{r^2 - x^2} = r^2 - 2x^2 \quad (\star) \quad \Rightarrow \quad \left(x\sqrt{r^2 - x^2}\,\right)^2 = (r^2 - 2x^2)^2 \quad \Rightarrow$

$x^2(r^2 - x^2) = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad r^2x^2 - x^4 = r^4 - 4r^2x^2 + 4x^4 \quad \Rightarrow \quad 5x^4 - 5r^2x^2 + r^4 = 0.$

This is a quadratic equation in x^2. By the quadratic formula, $x^2 = \frac{5 \pm \sqrt{5}}{10}r^2$, but we reject the solution with the + sign since it doesn't satisfy $(\star)$. [The right side is negative and the left side is positive.] So $x = \sqrt{\frac{5 - \sqrt{5}}{10}}\,r$. Since $S(0) = S(r) = 0$, the

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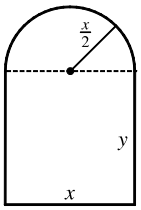
maximum surface area occurs at the critical number and $x^2 = \frac{5-\sqrt{5}}{10}r^2 \quad \Rightarrow \quad y^2 = r^2 - \frac{5-\sqrt{5}}{10}r^2 = \frac{5+\sqrt{5}}{10}r^2 \quad \Rightarrow$ the surface area is

$$2\pi\left(\tfrac{5+\sqrt{5}}{10}\right)r^2 + 4\pi\sqrt{\tfrac{5-\sqrt{5}}{10}}\sqrt{\tfrac{5+\sqrt{5}}{10}}r^2 = \pi r^2\left[2\cdot\tfrac{5+\sqrt{5}}{10} + 4\tfrac{\sqrt{(5-\sqrt{5})(5+\sqrt{5})}}{10}\right] = \pi r^2\left[\tfrac{5+\sqrt{5}}{5} + \tfrac{2\sqrt{20}}{5}\right]$$
$$= \pi r^2\left[\tfrac{5+\sqrt{5}+2\cdot 2\sqrt{5}}{5}\right] = \pi r^2\left[\tfrac{5+5\sqrt{5}}{5}\right] = \pi r^2\left(1+\sqrt{5}\right).$$

40.



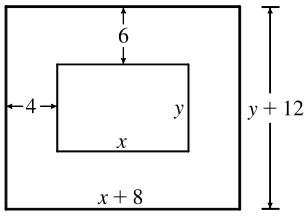
$\text{Perimeter} = 30 \quad \Rightarrow \quad 2y + x + \pi\left(\frac{x}{2}\right) = 30 \quad \Rightarrow$

$y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right) = 15 - \frac{x}{2} - \frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so $A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \quad \Rightarrow \quad x = \dfrac{15}{1+\pi/4} = \dfrac{60}{4+\pi}$. $A''(x) = -\left(1 + \dfrac{\pi}{4}\right) < 0$, so this gives a maximum.

The dimensions are $x = \dfrac{60}{4+\pi}$ ft and $y = 15 - \dfrac{30}{4+\pi} - \dfrac{15\pi}{4+\pi} = \dfrac{60+15\pi-30-15\pi}{4+\pi} = \dfrac{30}{4+\pi}$ ft, so the height of the rectangle is half the base.

41.



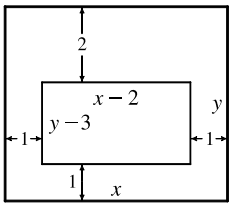
$xy = 384 \quad \Rightarrow \quad y = 384/x$. Total area is

$A(x) = (8+x)(12+384/x) = 12(40 + x + 256/x)$, so

$A'(x) = 12(1 - 256/x^2) = 0 \quad \Rightarrow \quad x = 16$. There is an absolute minimum when $x = 16$ since $A'(x) < 0$ for $0 < x < 16$ and $A'(x) > 0$ for $x > 16$.

When $x = 16$, $y = 384/16 = 24$, so the dimensions are 24 cm and 36 cm.

42.

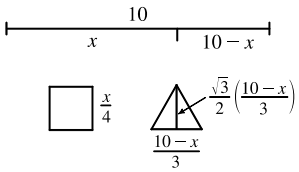


$xy = 180$, so $y = 180/x$. The printed area is

$(x-2)(y-3) = (x-2)(180/x - 3) = 186 - 3x - 360/x = A(x)$.

$A'(x) = -3 + 360/x^2 = 0$ when $x^2 = 120 \quad \Rightarrow \quad x = 2\sqrt{30}$. This gives an absolute maximum since $A'(x) > 0$ for $0 < x < 2\sqrt{30}$ and $A'(x) < 0$ for $x > 2\sqrt{30}$. When $x = 2\sqrt{30}$, $y = 180/(2\sqrt{30})$, so the dimensions are $2\sqrt{30}$ in. and $90/\sqrt{30}$ in.

43.



Let x be the length of the wire used for the square. The total area is

$$A(x) = \left(\frac{x}{4}\right)^2 + \frac{1}{2}\left(\frac{10-x}{3}\right)\frac{\sqrt{3}}{2}\left(\frac{10-x}{3}\right)$$
$$= \tfrac{1}{16}x^2 + \tfrac{\sqrt{3}}{36}(10-x)^2, \ 0 \le x \le 10$$

$A'(x) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) = 0 \quad \Leftrightarrow \quad \frac{9}{72}x + \frac{4\sqrt{3}}{72}x - \frac{40\sqrt{3}}{72} = 0 \quad \Leftrightarrow \quad x = \frac{40\sqrt{3}}{9+4\sqrt{3}}$.

Now $A(0) = \left(\frac{\sqrt{3}}{36}\right)100 \approx 4.81$, $A(10) = \frac{100}{16} = 6.25$ and $A\left(\frac{40\sqrt{3}}{9+4\sqrt{3}}\right) \approx 2.72$, so

(a) The maximum area occurs when $x = 10$ m, and all the wire is used for the square.

(b) The minimum area occurs when $x = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.35$ m.

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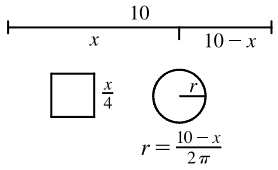
44.

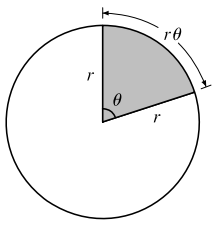


Total area is $A(x) = \left(\frac{x}{4}\right)^2 + \pi\left(\frac{10-x}{2\pi}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{4\pi},\ 0 \le x \le 10.$

$A'(x) = \frac{x}{8} - \frac{10-x}{2\pi} = \left(\frac{1}{2\pi} + \frac{1}{8}\right)x - \frac{5}{\pi} = 0 \quad \Rightarrow \quad x = 40/(4+\pi).$

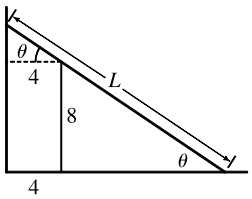
$A(0) = 25/\pi \approx 7.96$, $A(10) = 6.25$, and $A(40/(4+\pi)) \approx 3.5$, so the maximum occurs when $x = 0$ m and the minimum occurs when $x = 40/(4+\pi)$ m.

45.



From the figure, the perimeter of the slice is $2r + r\theta = 32$, so $\theta = \dfrac{32-2r}{r}$. The area A of the slice is $A = \frac{1}{2}r^2\theta = \frac{1}{2}r^2\left(\dfrac{32-2r}{r}\right) = r(16-r) = 16r - r^2$ for $0 \le r \le 16$. $A'(r) = 16 - 2r$, so $A' = 0$ when $r = 8$. Since $A(0) = 0$, $A(16) = 0$, and $A(8) = 64$ in.^{2}, the largest piece comes from a pizza with radius 8 in. and diameter 16 in. Note that $\theta = 2$ radians $\approx 114.6°$, which is about 32% of the whole pizza.

46.

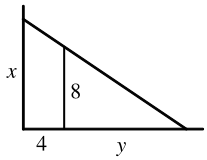


$L = 8\csc\theta + 4\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ \dfrac{dL}{d\theta} = -8\csc\theta\cot\theta + 4\sec\theta\tan\theta = 0$ when

$\sec\theta\tan\theta = 2\csc\theta\cot\theta \quad \Leftrightarrow \quad \tan^3\theta = 2 \quad \Leftrightarrow \quad \tan\theta = \sqrt[3]{2} \quad \Leftrightarrow \quad \theta = \tan^{-1}\sqrt[3]{2}.$

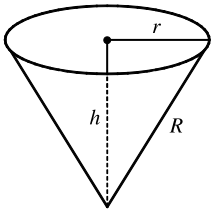
$dL/d\theta < 0$ when $0 < \theta < \tan^{-1}\sqrt[3]{2}$, $dL/d\theta > 0$ when $\tan^{-1}\sqrt[3]{2} < \theta < \frac{\pi}{2}$, so L has an absolute minimum when $\theta = \tan^{-1}\sqrt[3]{2}$, and the shortest ladder has length

$L = 8\dfrac{\sqrt{1+2^{2/3}}}{2^{1/3}} + 4\sqrt{1+2^{2/3}} \approx 16.65$ ft.



Another method: Minimize $L^2 = x^2 + (4+y)^2$, where $\dfrac{x}{4+y} = \dfrac{8}{y}$.

47.



$h^2 + r^2 = R^2 \quad \Rightarrow \quad V = \frac{\pi}{3}r^2h = \frac{\pi}{3}(R^2 - h^2)h = \frac{\pi}{3}(R^2h - h^3).$

$V'(h) = \frac{\pi}{3}(R^2 - 3h^2) = 0$ when $h = \frac{1}{\sqrt{3}}R$. This gives an absolute maximum, since $V'(h) > 0$ for $0 < h < \frac{1}{\sqrt{3}}R$ and $V'(h) < 0$ for $h > \frac{1}{\sqrt{3}}R$. The maximum volume is

$V\left(\frac{1}{\sqrt{3}}R\right) = \frac{\pi}{3}\left(\frac{1}{\sqrt{3}}R^3 - \frac{1}{3\sqrt{3}}R^3\right) = \frac{2}{9\sqrt{3}}\pi R^3.$

48. The volume and surface area of a cone with radius r and height h are given by $V = \frac{1}{3}\pi r^2h$ and $S = \pi r\sqrt{r^2+h^2}$.

We'll minimize $A = S^2$ subject to $V = 27$. $V = 27 \quad \Rightarrow \quad \frac{1}{3}\pi r^2h = 27 \quad \Rightarrow \quad r^2 = \dfrac{81}{\pi h}$ **(1)**.

$A = \pi^2r^2(r^2+h^2) = \pi^2\left(\dfrac{81}{\pi h}\right)\left(\dfrac{81}{\pi h} + h^2\right) = \dfrac{81^2}{h^2} + 81\pi h$, so $A' = 0 \quad \Rightarrow \quad \dfrac{-2\cdot 81^2}{h^3} + 81\pi = 0 \quad \Rightarrow$

$81\pi = \dfrac{2\cdot 81^2}{h^3} \quad \Rightarrow \quad h^3 = \dfrac{162}{\pi} \quad \Rightarrow \quad h = \sqrt[3]{\dfrac{162}{\pi}} = 3\sqrt[3]{\dfrac{6}{\pi}} \approx 3.722$. From **(1)**, $r^2 = \dfrac{81}{\pi h} = \dfrac{81}{\pi\cdot 3\sqrt[3]{6/\pi}} = \dfrac{27}{\sqrt[3]{6\pi^2}} \quad \Rightarrow$

$r = \dfrac{3\sqrt{3}}{\sqrt[6]{6\pi^2}} \approx 2.632$. $A'' = 6\cdot 81^2/h^4 > 0$, so A and hence S has an absolute minimum at these values of r and h.

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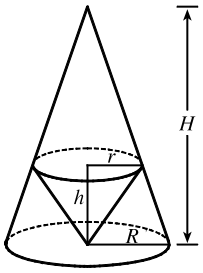
49.



By similar triangles, $\dfrac{H}{R}=\dfrac{H-h}{r}$ **(1)**. The volume of the inner cone is $V=\frac{1}{3}\pi r^2h$, so we'll solve **(1)** for h. $\dfrac{Hr}{R}=H-h \;\Rightarrow$

$h=H-\dfrac{Hr}{R}=\dfrac{HR-Hr}{R}=\dfrac{H}{R}(R-r)$ **(2)**.

Thus, $V(r)=\dfrac{\pi}{3}r^2\cdot\dfrac{H}{R}(R-r)=\dfrac{\pi H}{3R}(Rr^2-r^3) \;\Rightarrow$

$V'(r)=\dfrac{\pi H}{3R}(2Rr-3r^2)=\dfrac{\pi H}{3R}r(2R-3r)$.

$V'(r)=0 \;\Rightarrow\; r=0$ or $2R=3r \;\Rightarrow\; r=\frac{2}{3}R$ and from **(2)**, $h=\dfrac{H}{R}\left(R-\frac{2}{3}R\right)=\dfrac{H}{R}\left(\frac{1}{3}R\right)=\frac{1}{3}H$.

$V'(r)$ changes from positive to negative at $r=\frac{2}{3}R$, so the inner cone has a maximum volume of $V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi\left(\frac{2}{3}R\right)^2\left(\frac{1}{3}H\right)=\frac{4}{27}\cdot\frac{1}{3}\pi R^2H$, which is approximately 15% of the volume of the larger cone.

50. We need to minimize F for $0\le\theta<\pi/2$. $F(\theta)=\dfrac{\mu W}{\mu\sin\theta+\cos\theta} \;\Rightarrow\; F'(\theta)=\dfrac{-\mu W\,(\mu\cos\theta-\sin\theta)}{(\mu\sin\theta+\cos\theta)^2}$ [by the Reciprocal Rule]. $F'(\theta)>0 \;\Rightarrow\; \mu\cos\theta-\sin\theta<0 \;\Rightarrow\; \mu\cos\theta<\sin\theta \;\Rightarrow\; \mu<\tan\theta \;\Rightarrow\; \theta>\tan^{-1}\mu$. So F is decreasing on $\left(0,\tan^{-1}\mu\right)$ and increasing on $\left(\tan^{-1}\mu,\frac{\pi}{2}\right)$. Thus, F attains its minimum value at $\theta=\tan^{-1}\mu$. This maximum value is $F(\tan^{-1}\mu)=\dfrac{\mu W}{\sqrt{\mu^2+1}}$.

51. $P(R)=\dfrac{E^2R}{(R+r)^2} \;\Rightarrow$

$$P'(R)=\frac{(R+r)^2\cdot E^2-E^2R\cdot 2(R+r)}{[(R+r)^2]^2}=\frac{(R^2+2Rr+r^2)E^2-2E^2R^2-2E^2Rr}{(R+r)^4}$$

$$=\frac{E^2r^2-E^2R^2}{(R+r)^4}=\frac{E^2(r^2-R^2)}{(R+r)^4}=\frac{E^2(r+R)(r-R)}{(R+r)^4}=\frac{E^2(r-R)}{(R+r)^3}$$

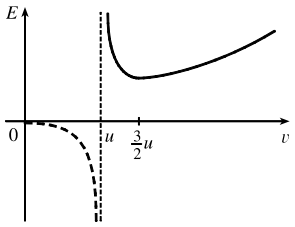
$P'(R)=0 \;\Rightarrow\; R=r \;\Rightarrow\; P(r)=\dfrac{E^2r}{(r+r)^2}=\dfrac{E^2r}{4r^2}=\dfrac{E^2}{4r}$.

The expression for $P'(R)$ shows that $P'(R)>0$ for $R<r$ and $P'(R)<0$ for $R>r$. Thus, the maximum value of the power is $E^2/(4r)$, and this occurs when $R=r$.

52. (a) $E(v)=\dfrac{aLv^3}{v-u} \;\Rightarrow\; E'(v)=aL\dfrac{(v-u)3v^2-v^3}{(v-u)^2}=0$ when $2v^3=3uv^2 \;\Rightarrow\; 2v=3u \;\Rightarrow\; v=\frac{3}{2}u$.

The First Derivative Test shows that this value of v gives the minimum value of E.

(b)



53. $S=6sh-\frac{3}{2}s^2\cot\theta+\left(\frac{3}{2}\sqrt{3}\,s^2\right)\csc\theta$

(a) $\dfrac{dS}{d\theta}=\frac{3}{2}s^2\csc^2\theta-\left(\frac{3}{2}\sqrt{3}\,s^2\right)\csc\theta\,\cot\theta$ or $\frac{3}{2}s^2\csc\theta\left(\csc\theta-\sqrt{3}\cot\theta\right)$.

(b) $\dfrac{dS}{d\theta}=0$ when $\csc\theta-\sqrt{3}\cot\theta=0 \;\Rightarrow\; \dfrac{1}{\sin\theta}-\sqrt{3}\,\dfrac{\cos\theta}{\sin\theta}=0 \;\Rightarrow\; \cos\theta=\frac{1}{\sqrt{3}}$. The First Derivative Test shows that the minimum surface area occurs when $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)\approx 55^\circ$.

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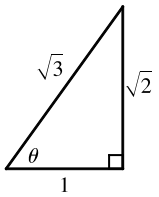
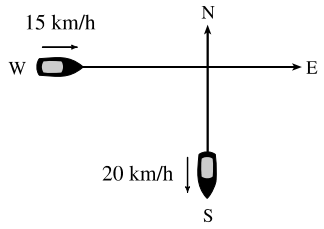
(c)



If $\cos\theta = \frac{1}{\sqrt{3}}$, then $\cot\theta = \frac{1}{\sqrt{2}}$ and $\csc\theta = \frac{\sqrt{3}}{\sqrt{2}}$, so the surface area is

$$S = 6sh - \tfrac{3}{2}s^2\tfrac{1}{\sqrt{2}} + 3s^2\tfrac{\sqrt{3}}{2}\tfrac{\sqrt{3}}{\sqrt{2}} = 6sh - \tfrac{3}{2\sqrt{2}}s^2 + \tfrac{9}{2\sqrt{2}}s^2$$
$$= 6sh + \tfrac{6}{2\sqrt{2}}s^2 = 6s\left(h + \tfrac{1}{2\sqrt{2}}s\right)$$

54.



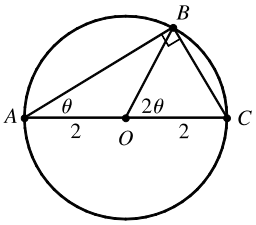
Let t be the time, in hours, after 2:00 PM. The position of the boat heading south at time t is $(0, -20t)$. The position of the boat heading east at time t is $(-15 + 15t, 0)$. If $D(t)$ is the distance between the boats at time t, we minimize $f(t) = [D(t)]^2 = 20^2t^2 + 15^2(t-1)^2$.

$f'(t) = 800t + 450(t-1) = 1250t - 450 = 0$ when $t = \frac{450}{1250} = 0.36$ h.

0.36 h $\times \frac{60 \text{ min}}{\text{h}} = 21.6$ min $= 21$ min 36 s. Since $f''(t) > 0$, this gives a minimum, so the boats are closest together at 2:21:36 PM.

55. Here $T(x) = \dfrac{\sqrt{x^2+25}}{6} + \dfrac{5-x}{8}$, $0 \le x \le 5$ $\Rightarrow$ $T'(x) = \dfrac{x}{6\sqrt{x^2+25}} - \dfrac{1}{8} = 0$ $\Leftrightarrow$ $8x = 6\sqrt{x^2+25}$ $\Leftrightarrow$ $16x^2 = 9(x^2+25)$ $\Leftrightarrow$ $x = \frac{15}{\sqrt{7}}$. But $\frac{15}{\sqrt{7}} > 5$, so T has no critical number. Since $T(0) \approx 1.46$ and $T(5) \approx 1.18$, she should row directly to B.

56.



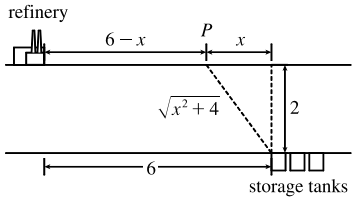
In isosceles triangle AOB, $\angle O = 180^\circ - \theta - \theta$, so $\angle BOC = 2\theta$. The distance rowed is $4\cos\theta$ while the distance walked is the length of arc $BC = 2(2\theta) = 4\theta$. The time taken is given by $T(\theta) = \dfrac{4\cos\theta}{2} + \dfrac{4\theta}{4} = 2\cos\theta + \theta$, $0 \le \theta \le \frac{\pi}{2}$.

$T'(\theta) = -2\sin\theta + 1 = 0$ $\Leftrightarrow$ $\sin\theta = \frac{1}{2}$ $\Rightarrow$ $\theta = \frac{\pi}{6}$.

Check the value of T at $\theta = \frac{\pi}{6}$ and at the endpoints of the domain of T; that is, $\theta = 0$ and $\theta = \frac{\pi}{2}$.

$T(0) = 2$, $T\left(\frac{\pi}{6}\right) = \sqrt{3} + \frac{\pi}{6} \approx 2.26$, and $T\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \approx 1.57$. Therefore, the minimum value of T is $\frac{\pi}{2}$ when $\theta = \frac{\pi}{2}$; that is, the woman should walk all the way. Note that $T''(\theta) = -2\cos\theta < 0$ for $0 \le \theta < \frac{\pi}{2}$, so $\theta = \frac{\pi}{6}$ gives a maximum time.

57. There are $(6-x)$ km over land and $\sqrt{x^2+4}$ km under the river.



We need to minimize the cost C (measured in \$100,000) of the pipeline.

$C(x) = (6-x)(4) + \left(\sqrt{x^2+4}\right)(8)$ $\Rightarrow$

$C'(x) = -4 + 8 \cdot \frac{1}{2}(x^2+4)^{-1/2}(2x) = -4 + \dfrac{8x}{\sqrt{x^2+4}}$.

$C'(x) = 0$ $\Rightarrow$ $4 = \dfrac{8x}{\sqrt{x^2+4}}$ $\Rightarrow$ $\sqrt{x^2+4} = 2x$ $\Rightarrow$ $x^2 + 4 = 4x^2$ $\Rightarrow$ $4 = 3x^2$ $\Rightarrow$ $x^2 = \frac{4}{3}$ $\Rightarrow$ $x = 2/\sqrt{3}$ $[0 \le x \le 6]$. Compare the costs for $x = 0$, $2/\sqrt{3}$, and 6. $C(0) = 24 + 16 = 40$,

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$C\big(2/\sqrt{3}\,\big) = 24 - 8/\sqrt{3} + 32/\sqrt{3} = 24 + 24/\sqrt{3} \approx 37.9$, and $C(6) = 0 + 8\sqrt{40} \approx 50.6$. So the minimum cost is about \$3.79 million when P is $6 - 2/\sqrt{3} \approx 4.85$ km east of the refinery.

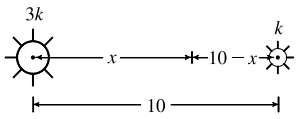
58. The distance from the refinery to P is now $\sqrt{(6-x)^2 + 1^2} = \sqrt{x^2 - 12x + 37}$.

Thus, $C(x) = 4\sqrt{x^2 - 12x + 37} + 8\sqrt{x^2 + 4} \;\Rightarrow$

$C'(x) = 4 \cdot \frac{1}{2}(x^2 - 12x + 37)^{-1/2}(2x - 12) + 8 \cdot \frac{1}{2}(x^2 + 4)^{-1/2}(2x) = \dfrac{4(x-6)}{\sqrt{x^2 - 12x + 37}} + \dfrac{8x}{\sqrt{x^2 + 4}}$.

$C'(x) = 0 \;\Rightarrow\; x \approx 1.12$ [from a graph of C' or a numerical rootfinder]. $C(0) \approx 40.3$, $C(1.12) \approx 38.3$, and $C(6) \approx 54.6$. So the minimum cost is slightly higher (than in the previous exercise) at about \$3.83 million when P is approximately 4.88 km from the point on the bank 1 km south of the refinery.

59.



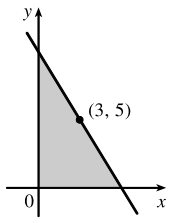
The total illumination is $I(x) = \dfrac{3k}{x^2} + \dfrac{k}{(10-x)^2},\ 0 < x < 10$. Then

$I'(x) = \dfrac{-6k}{x^3} + \dfrac{2k}{(10-x)^3} = 0 \;\Rightarrow\; 6k(10-x)^3 = 2kx^3 \;\Rightarrow$

$3(10-x)^3 = x^3 \;\Rightarrow\; \sqrt[3]{3}\,(10-x) = x \;\Rightarrow\; 10\sqrt[3]{3} - \sqrt[3]{3}\,x = x \;\Rightarrow\; 10\sqrt[3]{3} = x + \sqrt[3]{3}\,x \;\Rightarrow$

$10\sqrt[3]{3} = \big(1 + \sqrt[3]{3}\,\big)x \;\Rightarrow\; x = \dfrac{10\sqrt[3]{3}}{1 + \sqrt[3]{3}} \approx 5.9$ ft. This gives a minimum since $I''(x) > 0$ for $0 < x < 10$.

60.

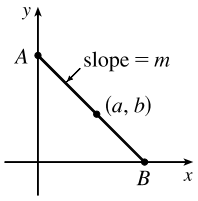


The line with slope m (where $m < 0$) through $(3, 5)$ has equation $y - 5 = m(x - 3)$ or $y = mx + (5 - 3m)$. The y-intercept is $5 - 3m$ and the x-intercept is $-5/m + 3$. So the triangle has area $A(m) = \frac{1}{2}(5 - 3m)(-5/m + 3) = 15 - 25/(2m) - \frac{9}{2}m$. Now

$A'(m) = \dfrac{25}{2m^2} - \dfrac{9}{2} = 0 \;\Leftrightarrow\; m^2 = \frac{25}{9} \;\Rightarrow\; m = -\frac{5}{3}$ (since $m < 0$).

$A''(m) = -\dfrac{25}{m^3} > 0$, so there is an absolute minimum when $m = -\frac{5}{3}$. Thus, an equation of the line is $y - 5 = -\frac{5}{3}(x - 3)$ or $y = -\frac{5}{3}x + 10$.

61.



Every line segment in the first quadrant passing through (a, b) with endpoints on the x- and y-axes satisfies an equation of the form $y - b = m(x - a)$, where $m < 0$. By setting $x = 0$ and then $y = 0$, we find its endpoints, $A(0, b - am)$ and $B\big(a - \frac{b}{m}, 0\big)$. The distance d from A to B is given by $d = \sqrt{[(a - \frac{b}{m}) - 0]^2 + [0 - (b - am)]^2}$.

It follows that the square of the length of the line segment, as a function of m, is given by

$S(m) = \left(a - \dfrac{b}{m}\right)^2 + (am - b)^2 = a^2 - \dfrac{2ab}{m} + \dfrac{b^2}{m^2} + a^2m^2 - 2abm + b^2$. Thus,

$$S'(m) = \frac{2ab}{m^2} - \frac{2b^2}{m^3} + 2a^2m - 2ab = \frac{2}{m^3}(abm - b^2 + a^2m^4 - abm^3)$$
$$= \frac{2}{m^3}[b(am - b) + am^3(am - b)] = \frac{2}{m^3}(am - b)(b + am^3)$$

Thus, $S'(m) = 0 \;\Leftrightarrow\; m = b/a$ or $m = -\sqrt[3]{\frac{b}{a}}$. Since $b/a > 0$ and $m < 0$, m must equal $-\sqrt[3]{\frac{b}{a}}$. Since $\dfrac{2}{m^3} < 0$, we see

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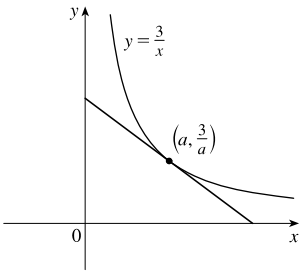
that $S'(m) < 0$ for $m < -\sqrt[3]{\frac{b}{a}}$ and $S'(m) > 0$ for $m > -\sqrt[3]{\frac{b}{a}}$. Thus, S has its absolute minimum value when $m = -\sqrt[3]{\frac{b}{a}}$. That value is

$$S\left(-\sqrt[3]{\tfrac{b}{a}}\right) = \left(a + b\sqrt[3]{\tfrac{a}{b}}\right)^2 + \left(-a\sqrt[3]{\tfrac{b}{a}} - b\right)^2 = \left(a + \sqrt[3]{ab^2}\right)^2 + \left(\sqrt[3]{a^2b} + b\right)^2$$
$$= a^2 + 2a^{4/3}b^{2/3} + a^{2/3}b^{4/3} + a^{4/3}b^{2/3} + 2a^{2/3}b^{4/3} + b^2 = a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2$$

The last expression is of the form $x^3 + 3x^2y + 3xy^2 + y^3 \quad [= (x+y)^3]$ with $x = a^{2/3}$ and $y = b^{2/3}$, so we can write it as $(a^{2/3} + b^{2/3})^3$ and the shortest such line segment has length $\sqrt{S} = (a^{2/3} + b^{2/3})^{3/2}$.

62. $y = 1 + 40x^3 - 3x^5 \quad \Rightarrow \quad y' = 120x^2 - 15x^4$, so the tangent line to the curve at $x = a$ has slope $m(a) = 120a^2 - 15a^4$. Now $m'(a) = 240a - 60a^3 = -60a(a^2 - 4) = -60a(a+2)(a-2)$, so $m'(a) > 0$ for $a < -2$, and $0 < a < 2$, and $m'(a) < 0$ for $-2 < a < 0$ and $a > 2$. Thus, m is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, increasing on $(0, 2)$, and decreasing on $(2, \infty)$. Clearly, $m(a) \to -\infty$ as $a \to \pm\infty$, so the maximum value of $m(a)$ must be one of the two local maxima, $m(-2)$ or $m(2)$. But both $m(-2)$ and $m(2)$ equal $120 \cdot 2^2 - 15 \cdot 2^4 = 480 - 240 = 240$. So 240 is the largest slope, and it occurs at the points $(-2, -223)$ and $(2, 225)$. *Note:* $a = 0$ corresponds to a local *minimum* of m.

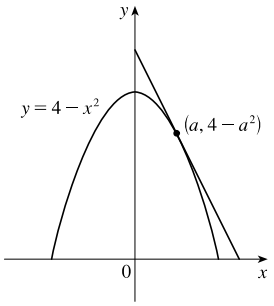
63. $y = \dfrac{3}{x} \quad \Rightarrow \quad y' = -\dfrac{3}{x^2}$, so an equation of the tangent line at the point $(a, \frac{3}{a})$ is $y - \dfrac{3}{a} = -\dfrac{3}{a^2}(x - a)$, or $y = -\dfrac{3}{a^2}x + \dfrac{6}{a}$. The y-intercept $\quad [x = 0] \quad$ is $6/a$. The x-intercept $\quad [y = 0] \quad$ is $2a$. The distance d of the line segment that has endpoints at the intercepts is $d = \sqrt{(2a-0)^2 + (0 - 6/a)^2}$. Let $S = d^2$, so $S = 4a^2 + \dfrac{36}{a^2} \quad \Rightarrow$



$S' = 8a - \dfrac{72}{a^3}$. $S' = 0 \quad \Leftrightarrow \quad \dfrac{72}{a^3} = 8a \quad \Leftrightarrow \quad a^4 = 9 \quad \Leftrightarrow \quad a^2 = 3 \quad \Rightarrow \quad a = \sqrt{3}$.

$S'' = 8 + \dfrac{216}{a^4} > 0$, so there is an absolute minimum at $a = \sqrt{3}$. Thus, $S = 4(3) + \frac{36}{3} = 12 + 12 = 24$ and hence, $d = \sqrt{24} = 2\sqrt{6}$.

64. $y = 4 - x^2 \quad \Rightarrow \quad y' = -2x$, so an equation of the tangent line at $(a, 4 - a^2)$ is $y - (4 - a^2) = -2a(x - a)$, or $y = -2ax + a^2 + 4$. The y-intercept $\quad [x = 0]$ is $a^2 + 4$. The x-intercept $\quad [y = 0] \quad$ is $\dfrac{a^2 + 4}{2a}$. The area A of the triangle is



$A = \dfrac{1}{2}(\text{base})(\text{height}) = \dfrac{1}{2} \cdot \dfrac{a^2 + 4}{2a}(a^2 + 4) = \dfrac{1}{4}\dfrac{a^4 + 8a^2 + 16}{a} = \dfrac{1}{4}\left(a^3 + 8a + \dfrac{16}{a}\right)$.

$A' = 0 \quad \Rightarrow \quad \dfrac{1}{4}\left(3a^2 + 8 - \dfrac{16}{a^2}\right) = 0 \quad \Rightarrow \quad 3a^4 + 8a^2 - 16 = 0 \quad \Rightarrow$

$(3a^2 - 4)(a^2 + 4) = 0 \quad \Rightarrow \quad a^2 = \dfrac{4}{3} \quad \Rightarrow \quad a = \dfrac{2}{\sqrt{3}}$. $A'' = \dfrac{1}{4}\left(6a + \dfrac{32}{a^3}\right) > 0$, so there is an absolute minimum at $a = \dfrac{2}{\sqrt{3}}$. Thus, $A = \dfrac{1}{2} \cdot \dfrac{4/3 + 4}{2(2/\sqrt{3})}\left(\dfrac{4}{3} + 4\right) = \dfrac{1}{2} \cdot \dfrac{4\sqrt{3}}{3} \cdot \dfrac{16}{3} = \dfrac{32}{9}\sqrt{3}$.

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65. (a) If $c(x) = \dfrac{C(x)}{x}$, then, by the Quotient Rule, we have $c'(x) = \dfrac{xC'(x) - C(x)}{x^2}$. Now $c'(x) = 0$ when $xC'(x) - C(x) = 0$ and this gives $C'(x) = \dfrac{C(x)}{x} = c(x)$. Therefore, the marginal cost equals the average cost.

(b) (i) $C(x) = 16{,}000 + 200x + 4x^{3/2}$, $C(1000) = 16{,}000 + 200{,}000 + 40{,}000\sqrt{10} \approx 216{,}000 + 126{,}491$, so $C(1000) \approx \$342{,}491$. $c(x) = C(x)/x = \dfrac{16{,}000}{x} + 200 + 4x^{1/2}$, $c(1000) \approx \$342.49/\text{unit}$. $C'(x) = 200 + 6x^{1/2}$, $C'(1000) = 200 + 60\sqrt{10} \approx \$389.74/\text{unit}$.

(ii) We must have $C'(x) = c(x) \quad \Leftrightarrow \quad 200 + 6x^{1/2} = \dfrac{16{,}000}{x} + 200 + 4x^{1/2} \quad \Leftrightarrow \quad 2x^{3/2} = 16{,}000 \quad \Leftrightarrow$ $x = (8{,}000)^{2/3} = 400$ units. To check that this is a minimum, we calculate $c'(x) = \dfrac{-16{,}000}{x^2} + \dfrac{2}{\sqrt{x}} = \dfrac{2}{x^2}\left(x^{3/2} - 8000\right)$. This is negative for $x < (8000)^{2/3} = 400$, zero at $x = 400$, and positive for $x > 400$, so c is decreasing on $(0, 400)$ and increasing on $(400, \infty)$. Thus, c has an absolute minimum at $x = 400$. [*Note:* $c''(x)$ is *not* positive for all $x > 0$.]

(iii) The minimum average cost is $c(400) = 40 + 200 + 80 = \$320/\text{unit}$.

66. (a) The total profit is $P(x) = R(x) - C(x)$. In order to maximize profit we look for the critical numbers of P, that is, the numbers where the marginal profit is 0. But if $P'(x) = R'(x) - C'(x) = 0$, then $R'(x) = C'(x)$. Therefore, if the profit is a maximum, then the marginal revenue equals the marginal cost.

(b) $C(x) = 16{,}000 + 500x - 1.6x^2 + 0.004x^3$, $p(x) = 1700 - 7x$. Then $R(x) = xp(x) = 1700x - 7x^2$. If the profit is maximum, then $R'(x) = C'(x) \quad \Leftrightarrow \quad 1700 - 14x = 500 - 3.2x + 0.012x^2 \quad \Leftrightarrow \quad 0.012x^2 + 10.8x - 1200 = 0 \quad \Leftrightarrow$ $x^2 + 900x - 100{,}000 = 0 \quad \Leftrightarrow \quad (x + 1000)(x - 100) = 0 \quad \Leftrightarrow \quad x = 100$ (since $x > 0$). The profit is maximized if $P''(x) < 0$, but since $P''(x) = R''(x) - C''(x)$, we can just check the condition $R''(x) < C''(x)$. Now $R''(x) = -14 < -3.2 + 0.024x = C''(x)$ for $x > 0$, so there is a maximum at $x = 100$.

67. (a) We are given that the demand function p is linear and $p(27{,}000) = 10$, $p(33{,}000) = 8$, so the slope is $\frac{10 - 8}{27{,}000 - 33{,}000} = -\frac{1}{3000}$ and an equation of the line is $y - 10 = \left(-\frac{1}{3000}\right)(x - 27{,}000) \quad \Rightarrow$ $y = p(x) = -\frac{1}{3000}x + 19 = 19 - (x/3000)$.

(b) The revenue is $R(x) = xp(x) = 19x - (x^2/3000) \quad \Rightarrow \quad R'(x) = 19 - (x/1500) = 0$ when $x = 28{,}500$. Since $R''(x) = -1/1500 < 0$, the maximum revenue occurs when $x = 28{,}500 \quad \Rightarrow \quad$ the price is $p(28{,}500) = \$9.50$.

68. (a) Let $p(x)$ be the demand function. Then $p(x)$ is linear and $y = p(x)$ passes through $(20, 10)$ and $(18, 11)$, so the slope is $-\frac{1}{2}$ and an equation of the line is $y - 10 = -\frac{1}{2}(x - 20) \quad \Leftrightarrow \quad y = -\frac{1}{2}x + 20$. Thus, the demand is $p(x) = -\frac{1}{2}x + 20$ and the revenue is $R(x) = xp(x) = -\frac{1}{2}x^2 + 20x$.

(b) The cost is $C(x) = 6x$, so the profit is $P(x) = R(x) - C(x) = -\frac{1}{2}x^2 + 14x$. Then $0 = P'(x) = -x + 14 \quad \Rightarrow$ $x = 14$. Since $P''(x) = -1 < 0$, the selling price for maximum profit is $p(14) = -\frac{1}{2}(14) + 20 = \$13$.

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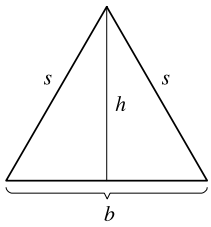
69. (a) As in Example 6, we see that the demand function p is linear. We are given that $p(1200) = 350$ and deduce that $p(1280) = 340$, since a \$10 reduction in price increases sales by 80 per week. The slope for p is $\dfrac{340-350}{1280-1200} = -\dfrac{1}{8}$, so an equation is $p - 350 = -\frac{1}{8}(x - 1200)$ or $p(x) = -\frac{1}{8}x + 500$, where $x \geq 1200$.

(b) $R(x) = x\,p(x) = -\frac{1}{8}x^2 + 500x$. $R'(x) = -\frac{1}{4}x + 500 = 0$ when $x = 4(500) = 2000$. $p(2000) = 250$, so the price should be set at \$250 to maximize revenue.

(c) $C(x) = 35{,}000 + 120x \;\Rightarrow\; P(x) = R(x) - C(x) = -\frac{1}{8}x^2 + 500x - 35{,}000 - 120x = -\frac{1}{8}x^2 + 380x - 35{,}000$. $P'(x) = -\frac{1}{4}x + 380 = 0$ when $x = 4(380) = 1520$. $p(1520) = 310$, so the price should be set at \$310 to maximize profit.

70. Let w denote the number of operating wells. Then the amount of daily oil production for each well is $240 - 8(w - 16) = 368 - 8w$, where $w \geq 16$. The total daily oil production P for all wells is given by $P(w) = w(368 - 8w) = 368w - 8w^2$. Now $P'(w) = 368 - 16w$ and $P'(w) = 0 \;\Leftrightarrow\; w = \frac{368}{16} = 23$. $P''(w) = -16 < 0$, so the daily production is maximized when the company adds $23 - 16 = 7$ wells.

71.



Here $s^2 = h^2 + b^2/4$, so $h^2 = s^2 - b^2/4$. The area is $A = \frac{1}{2}b\sqrt{s^2 - b^2/4}$. Let the perimeter be p, so $2s + b = p$ or $s = (p - b)/2 \;\Rightarrow$ $A(b) = \frac{1}{2}b\sqrt{(p-b)^2/4 - b^2/4} = b\sqrt{p^2 - 2pb}/4$. Now

$$A'(b) = \frac{\sqrt{p^2 - 2pb}}{4} - \frac{bp/4}{\sqrt{p^2 - 2pb}} = \frac{-3pb + p^2}{4\sqrt{p^2 - 2pb}}.$$

Therefore, $A'(b) = 0 \;\Rightarrow\; -3pb + p^2 = 0 \;\Rightarrow\; b = p/3$. Since $A'(b) > 0$ for $b < p/3$ and $A'(b) < 0$ for $b > p/3$, there is an absolute maximum when $b = p/3$. But then $2s + p/3 = p$, so $s = p/3 \;\Rightarrow\; s = b \;\Rightarrow$ the triangle is equilateral.

72. From Exercise 57, with K replacing 8 for the "under river" cost (measured in \$100,000), we see that $C'(x) = 0 \;\Leftrightarrow$ $4\sqrt{x^2+4} = Kx \;\Leftrightarrow\; 16x^2 + 64 = K^2x^2 \;\Leftrightarrow\; 64 = (K^2 - 16)x^2 \;\Leftrightarrow\; x = \dfrac{8}{\sqrt{K^2-16}}$. Also from Exercise 57, we have $C(x) = (6 - x)4 + \sqrt{x^2+4}\,K$. We now compare costs for using the minimum distance possible under the river [$x = 0$] and using the critical number above. $C(0) = 24 + 2K$ and

$$\begin{aligned} C\left(\frac{8}{\sqrt{K^2-16}}\right) &= 24 - \frac{32}{\sqrt{K^2-16}} + \sqrt{\frac{64}{K^2-16} + 4}\,K = 24 - \frac{32}{\sqrt{K^2-16}} + \sqrt{\frac{4K^2}{K^2-16}}\,K \\ &= 24 - \frac{32}{\sqrt{K^2-16}} + \frac{2K^2}{\sqrt{K^2-16}} = 24 + \frac{2(K^2-16)}{\sqrt{K^2-16}} = 24 + 2\sqrt{K^2-16} \end{aligned}$$

Since $\sqrt{K^2-16} < K$, we see that $C\left(\dfrac{8}{\sqrt{K^2-16}}\right) < C(0)$ for any cost K, so the minimum distance possible for the "under river" portion of the pipeline should *never* be used.

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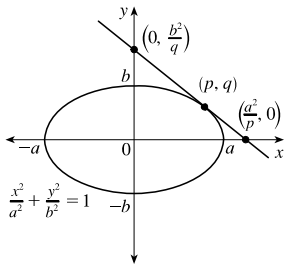
73. (a)



Using implicit differentiation, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{2x}{a^2} + \frac{2y\,y'}{b^2} = 0 \quad \Rightarrow$ $\frac{2y\,y'}{b^2} = -\frac{2x}{a^2} \quad \Rightarrow \quad y' = -\frac{b^2x}{a^2y}$. At (p, q), $y' = -\frac{b^2p}{a^2q}$, and an equation of the tangent line is $y - q = -\frac{b^2p}{a^2q}(x - p) \quad \Leftrightarrow \quad y = -\frac{b^2p}{a^2q}x + \frac{b^2p^2}{a^2q} + q \quad \Leftrightarrow$ $y = -\frac{b^2p}{a^2q}x + \frac{b^2p^2 + a^2q^2}{a^2q}$. The last term is the y-intercept, but not the term we want, namely b^2/q. Since (p, q) is on the ellipse, we know $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1$. To use that relationship we must divide b^2p^2 in the y-intercept by a^2b^2, so divide all terms by a^2b^2. $\frac{(b^2p^2 + a^2q^2)/a^2b^2}{(a^2q)/a^2b^2} = \frac{p^2/a^2 + q^2/b^2}{q/b^2} = \frac{1}{q/b^2} = \frac{b^2}{q}$. So the tangent line has equation $y = -\frac{b^2p}{a^2q}x + \frac{b^2}{q}$. Let $y = 0$ and solve for x to find that x-intercept: $\frac{b^2p}{a^2q}x = \frac{b^2}{q} \quad \Leftrightarrow$ $x = \frac{b^2a^2q}{qb^2p} = \frac{a^2}{p}$.

(b) The portion of the tangent line cut off by the coordinate axes is the distance between the intercepts, $\left(a^2/p, 0\right)$ and $\left(0, b^2/q\right)$: $\sqrt{\left(\frac{a^2}{p}\right)^2 + \left(-\frac{b^2}{q}\right)^2} = \sqrt{\frac{a^4}{p^2} + \frac{b^4}{q^2}}$. To eliminate p or q, we turn to the relationship $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \quad \Leftrightarrow$ $\frac{q^2}{b^2} = 1 - \frac{p^2}{a^2} \quad \Leftrightarrow \quad q^2 = b^2 - \frac{b^2p^2}{a^2} \quad \Leftrightarrow \quad q^2 = \frac{b^2(a^2 - p^2)}{a^2}$. Now substitute for q^2 and use the square S of the distance. $S(p) = \frac{a^4}{p^2} + \frac{b^4a^2}{b^2(a^2 - p^2)} = \frac{a^4}{p^2} + \frac{a^2b^2}{a^2 - p^2}$ for $0 < p < a$. Note that as $p \to 0$ or $p \to a$, $S(p) \to \infty$, so the minimum value of S must occur at a critical number. Now $S'(p) = -\frac{2a^4}{p^3} + \frac{2a^2b^2p}{(a^2 - p^2)^2}$ and $S'(p) = 0 \quad \Leftrightarrow$ $\frac{2a^4}{p^3} = \frac{2a^2b^2p}{(a^2 - p^2)^2} \quad \Leftrightarrow \quad a^2(a^2 - p^2)^2 = b^2p^4 \quad \Rightarrow \quad a(a^2 - p^2) = bp^2 \quad \Leftrightarrow \quad a^3 = (a + b)p^2 \quad \Leftrightarrow \quad p^2 = \frac{a^3}{a + b}$. Substitute for p^2 in $S(p)$:

$$\frac{a^4}{\frac{a^3}{a+b}} + \frac{a^2b^2}{a^2 - \frac{a^3}{a+b}} = \frac{a^4(a+b)}{a^3} + \frac{a^2b^2(a+b)}{a^2(a+b) - a^3} = \frac{a(a+b)}{1} + \frac{a^2b^2(a+b)}{a^2b}$$
$$= a(a+b) + b(a+b) = (a+b)(a+b) = (a+b)^2$$

Taking the square root gives us the desired minimum length of $a + b$.

(c) The triangle formed by the tangent line and the coordinate axes has area $A = \frac{1}{2}\left(\frac{a^2}{p}\right)\left(\frac{b^2}{q}\right)$. As in part (b), we'll use the square of the area and substitute for q^2. $S = \frac{a^4b^4}{4p^2q^2} = \frac{a^4b^4a^2}{4p^2b^2(a^2 - p^2)} = \frac{a^6b^2}{4p^2(a^2 - p^2)}$. Minimizing S (and hence A) is equivalent to maximizing $p^2(a^2 - p^2)$. Let $f(p) = p^2(a^2 - p^2) = a^2p^2 - p^4$ for $0 < p < a$. As in part (b), the

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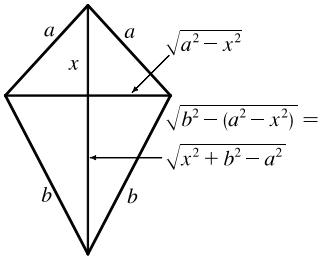
minimum value of S must occur at a critical number. Now $f'(p) = 2a^2p - 4p^3 = 2p(a^2 - 2p^2)$. $f'(p) = 0 \Rightarrow$ $p^2 = a^2/2 \Rightarrow p = a/\sqrt{2}$ $[p > 0]$. Substitute for p^2 in $S(p)$: $\dfrac{a^6b^2}{4\left(\frac{a^2}{2}\right)\left(a^2 - \frac{a^2}{2}\right)} = \dfrac{a^6b^2}{a^4} = a^2b^2 = (ab)^2$.

Taking the square root gives us the desired minimum area of ab.

74. See the figure. The area is given by



$$A(x) = \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\,\right)x + \tfrac{1}{2}\left(2\sqrt{a^2 - x^2}\,\right)\left(\sqrt{x^2 + b^2 - a^2}\,\right)$$
$$= \sqrt{a^2 - x^2}\left(x + \sqrt{x^2 + b^2 - a^2}\,\right)$$

for $0 \le x \le a$. Now

$$A'(x) = \sqrt{a^2 - x^2}\left(1 + \frac{x}{\sqrt{x^2 + b^2 - a^2}}\right) + \left(x + \sqrt{x^2 + b^2 - a^2}\,\right)\frac{-x}{\sqrt{a^2 - x^2}}$$
$$= 0 \quad \Leftrightarrow$$
$$\frac{x}{\sqrt{a^2 - x^2}}\left(x + \sqrt{x^2 + b^2 - a^2}\,\right) = \sqrt{a^2 - x^2}\left(\frac{x + \sqrt{x^2 + b^2 - a^2}}{\sqrt{x^2 + b^2 - a^2}}\right).$$

Except for the trivial case where $x = 0$, $a = b$ and $A(x) = 0$, we have $x + \sqrt{x^2 + b^2 - a^2} > 0$. Hence, cancelling this factor gives $\dfrac{x}{\sqrt{a^2 - x^2}} = \dfrac{\sqrt{a^2 - x^2}}{\sqrt{x^2 + b^2 - a^2}} \Rightarrow x\sqrt{x^2 + b^2 - a^2} = a^2 - x^2 \Rightarrow$

$x^2(x^2 + b^2 - a^2) = a^4 - 2a^2x^2 + x^4 \Rightarrow x^2(b^2 - a^2) = a^4 - 2a^2x^2 \Rightarrow x^2(b^2 + a^2) = a^4 \Rightarrow x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$.

Now we must check the value of A at this point as well as at the endpoints of the domain to see which gives the maximum value. $A(0) = a\sqrt{b^2 - a^2}$, $A(a) = 0$ and

$$A\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right) = \sqrt{a^2 - \left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2}\left[\frac{a^2}{\sqrt{a^2 + b^2}} + \sqrt{\left(\frac{a^2}{\sqrt{a^2 + b^2}}\right)^2 + b^2 - a^2}\right]$$
$$= \frac{ab}{\sqrt{a^2 + b^2}}\left[\frac{a^2}{\sqrt{a^2 + b^2}} + \frac{b^2}{\sqrt{a^2 + b^2}}\right] = \frac{ab(a^2 + b^2)}{a^2 + b^2} = ab$$

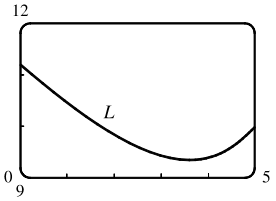
Since $b \ge \sqrt{b^2 - a^2}$, $A\left(a^2/\sqrt{a^2 + b^2}\right) \ge A(0)$. So there is an absolute maximum when $x = \dfrac{a^2}{\sqrt{a^2 + b^2}}$. In this case the horizontal piece should be $\dfrac{2ab}{\sqrt{a^2 + b^2}}$ and the vertical piece should be $\dfrac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2}$.

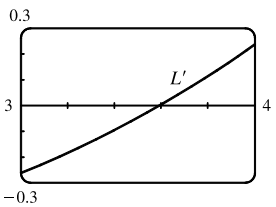
75. Note that $|AD| = |AP| + |PD| \Rightarrow 5 = x + |PD| \Rightarrow |PD| = 5 - x$.

Using the Pythagorean Theorem for ΔPDB and ΔPDC gives us

$$L(x) = |AP| + |BP| + |CP| = x + \sqrt{(5 - x)^2 + 2^2} + \sqrt{(5 - x)^2 + 3^2}$$
$$= x + \sqrt{x^2 - 10x + 29} + \sqrt{x^2 - 10x + 34} \Rightarrow$$

$L'(x) = 1 + \dfrac{x - 5}{\sqrt{x^2 - 10x + 29}} + \dfrac{x - 5}{\sqrt{x^2 - 10x + 34}}$. From the graphs of L and L', it seems that the minimum value of L is about $L(3.59) = 9.35$ m.





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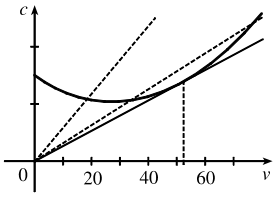
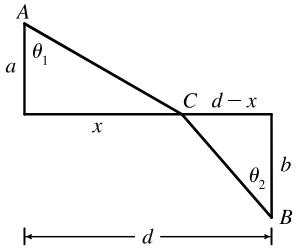
76. We note that since c is the consumption in gallons per hour, and v is the velocity in miles per hour, then $\dfrac{c}{v} = \dfrac{\text{gallons/hour}}{\text{miles/hour}} = \dfrac{\text{gallons}}{\text{mile}}$ gives us the consumption in gallons per mile, that is, the quantity G. To find the minimum, we calculate $\dfrac{dG}{dv} = \dfrac{d}{dv}\left(\dfrac{c}{v}\right) = \dfrac{v\,\dfrac{dc}{dv} - c\,\dfrac{dv}{dv}}{v^2} = \dfrac{v\,\dfrac{dc}{dv} - c}{v^2}$.



This is 0 when $v\,\dfrac{dc}{dv} - c = 0 \quad \Leftrightarrow \quad \dfrac{dc}{dv} = \dfrac{c}{v}$. This implies that the tangent line of $c(v)$ passes through the origin, and this occurs when $v \approx 53$ mi/h. Note that the slope of the secant line through the origin and a point $(v, c(v))$ on the graph is equal to $G(v)$, and it is intuitively clear that G is minimized in the case where the secant is in fact a tangent.

77.



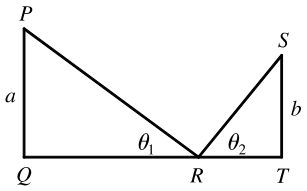
The total time is

$$T(x) = (\text{time from } A \text{ to } C) + (\text{time from } C \text{ to } B)$$
$$= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2},\ 0 < x < d$$

$$T'(x) = \frac{x}{v_1\sqrt{a^2 + x^2}} - \frac{d-x}{v_2\sqrt{b^2 + (d-x)^2}} = \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2}$$

The minimum occurs when $T'(x) = 0 \quad \Rightarrow \quad \dfrac{\sin\theta_1}{v_1} = \dfrac{\sin\theta_2}{v_2}$, or, equivalently, $\dfrac{\sin\theta_1}{\sin\theta_2} = \dfrac{v_1}{v_2}$. [*Note:* $T''(x) > 0$]

78.



If $d = |QT|$, we minimize $f(\theta_1) = |PR| + |RS| = a\csc\theta_1 + b\csc\theta_2$. Differentiating with respect to θ_1, and setting $\dfrac{df}{d\theta_1}$ equal to 0, we get

$$\frac{df}{d\theta_1} = 0 = -a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\,\frac{d\theta_2}{d\theta_1}.$$

So we need to find an expression for $\dfrac{d\theta_2}{d\theta_1}$. We can do this by observing that $|QT| = \text{constant} = a\cot\theta_1 + b\cot\theta_2$.

Differentiating this equation implicitly with respect to θ_1, we get $-a\csc^2\theta_1 - b\csc^2\theta_2\,\dfrac{d\theta_2}{d\theta_1} = 0 \quad \Rightarrow$

$\dfrac{d\theta_2}{d\theta_1} = -\dfrac{a\csc^2\theta_1}{b\csc^2\theta_2}$. We substitute this into the expression for $\dfrac{df}{d\theta_1}$ to get

$$-a\csc\theta_1\,\cot\theta_1 - b\csc\theta_2\,\cot\theta_2\left(-\frac{a\csc^2\theta_1}{b\csc^2\theta_2}\right) = 0 \quad \Leftrightarrow \quad -a\csc\theta_1\,\cot\theta_1 + a\frac{\csc^2\theta_1\,\cot\theta_2}{\csc\theta_2} = 0 \quad \Leftrightarrow$$

$\cot\theta_1\,\csc\theta_2 = \csc\theta_1\,\cot\theta_2 \quad \Leftrightarrow \quad \dfrac{\cot\theta_1}{\csc\theta_1} = \dfrac{\cot\theta_2}{\csc\theta_2} \quad \Leftrightarrow \quad \cos\theta_1 = \cos\theta_2$. Since θ_1 and θ_2 are both acute, we have $\theta_1 = \theta_2$.

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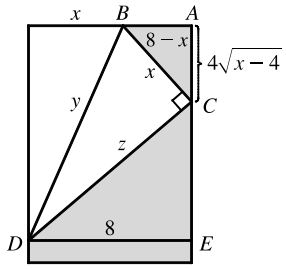
79.

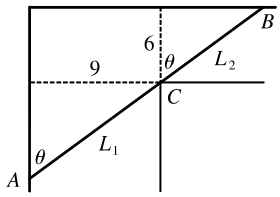


$y^2 = x^2 + z^2$, but triangles CDE and BCA are similar, so $z/8 = x/\left(4\sqrt{x-4}\right) \quad \Rightarrow \quad z = 2x/\sqrt{x-4}$. Thus, we minimize $f(x) = y^2 = x^2 + 4x^2/(x-4) = x^3/(x-4),\ 4 < x \le 8$.

$$f'(x) = \frac{(x-4)(3x^2) - x^3}{(x-4)^2} = \frac{x^2[3(x-4)-x]}{(x-4)^2} = \frac{2x^2(x-6)}{(x-4)^2} = 0$$

when $x = 6$. $f'(x) < 0$ when $x < 6$, $f'(x) > 0$ when $x > 6$, so the minimum occurs when $x = 6$ in.

80.

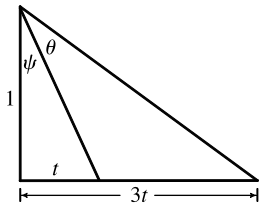


Paradoxically, we solve this maximum problem by solving a minimum problem. Let L be the length of the line ACB going from wall to wall touching the inner corner C. As $\theta \to 0$ or $\theta \to \frac{\pi}{2}$, we have $L \to \infty$ and there will be an angle that makes L a minimum. A pipe of this length will just fit around the corner.

From the diagram, $L = L_1 + L_2 = 9\csc\theta + 6\sec\theta \quad \Rightarrow \quad dL/d\theta = -9\csc\theta\,\cot\theta + 6\sec\theta\,\tan\theta = 0$ when $6\sec\theta\,\tan\theta = 9\csc\theta\,\cot\theta \quad \Leftrightarrow \quad \tan^3\theta = \frac{9}{6} = 1.5 \quad \Leftrightarrow \quad \tan\theta = \sqrt[3]{1.5}$. Then $\sec^2\theta = 1 + \left(\frac{3}{2}\right)^{2/3}$ and $\csc^2\theta = 1 + \left(\frac{3}{2}\right)^{-2/3}$, so the longest pipe has length $L = 9\left[1 + \left(\frac{3}{2}\right)^{-2/3}\right]^{1/2} + 6\left[1 + \left(\frac{3}{2}\right)^{2/3}\right]^{1/2} \approx 21.07$ ft.

Or, use $\theta = \tan^{-1}\left(\sqrt[3]{1.5}\right) \approx 0.853 \quad \Rightarrow \quad L = 9\csc\theta + 6\sec\theta \approx 21.07$ ft.

81.

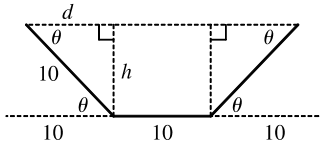


$$\theta = (\theta + \psi) - \psi = \arctan\frac{3t}{1} - \arctan\frac{t}{1} \quad \Rightarrow \quad \theta' = \frac{3}{1+9t^2} - \frac{1}{1+t^2}.$$

$\theta' = 0 \quad \Rightarrow \quad \frac{3}{1+9t^2} = \frac{1}{1+t^2} \quad \Rightarrow \quad 3 + 3t^2 = 1 + 9t^2 \quad \Rightarrow \quad 2 = 6t^2 \quad \Rightarrow$ $t^2 = \frac{1}{3} \quad \Rightarrow \quad t = 1/\sqrt{3}$. Thus,

$\theta = \arctan 3/\sqrt{3} - \arctan 1/\sqrt{3} = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$.

82. We maximize the cross-sectional area



$$A(\theta) = 10h + 2\left(\tfrac{1}{2}dh\right) = 10h + dh = 10(10\sin\theta) + (10\cos\theta)(10\sin\theta)$$
$$= 100(\sin\theta + \sin\theta\,\cos\theta),\ 0 \le \theta \le \tfrac{\pi}{2}$$
$$A'(\theta) = 100(\cos\theta + \cos^2\theta - \sin^2\theta) = 100(\cos\theta + 2\cos^2\theta - 1)$$
$$= 100(2\cos\theta - 1)(\cos\theta + 1) = 0 \text{ when } \cos\theta = \tfrac{1}{2} \quad \Leftrightarrow \quad \theta = \tfrac{\pi}{3} \qquad [\cos\theta \ne -1 \text{ since } 0 \le \theta \le \tfrac{\pi}{2}.]$$

Now $A(0) = 0$, $A\left(\frac{\pi}{2}\right) = 100$ and $A\left(\frac{\pi}{3}\right) = 75\sqrt{3} \approx 129.9$, so the maximum occurs when $\theta = \frac{\pi}{3}$.

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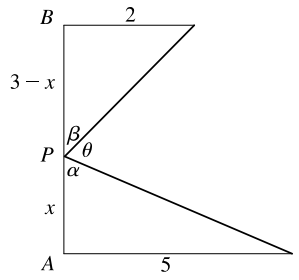
83.



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

$\alpha + \beta + \theta = 180^\circ = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \quad \Rightarrow$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\dfrac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\dfrac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\dfrac{d\theta}{dx} = 0 \quad \Rightarrow \quad \dfrac{5}{x^2+25} = \dfrac{2}{x^2-6x+13} \quad \Rightarrow \quad 2x^2+50 = 5x^2-30x+65 \quad \Rightarrow$

$3x^2 - 30x + 15 = 0 \quad \Rightarrow \quad x^2 - 10x + 5 = 0 \quad \Rightarrow \quad x = 5 \pm 2\sqrt{5}$. We reject the solution with the $+$ sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

84. Let x be the distance from the observer to the wall. Then, from the text figure,

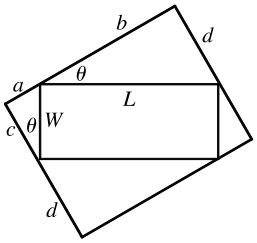
$\theta = \tan^{-1}\left(\dfrac{h+d}{x}\right) - \tan^{-1}\left(\dfrac{d}{x}\right), \quad x > 0 \quad \Rightarrow$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{d[x^2+(h+d)^2] - (h+d)(x^2+d^2)}{[x^2+(h+d)^2](x^2+d^2)} = \frac{h^2d + hd^2 - hx^2}{[x^2+(h+d)^2](x^2+d^2)} = 0 \quad \Leftrightarrow$$

$hx^2 = h^2d + hd^2 \quad \Leftrightarrow \quad x^2 = hd + d^2 \quad \Leftrightarrow \quad x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

85.



In the small triangle with sides a and c and hypotenuse W, $\sin\theta = \dfrac{a}{W}$ and $\cos\theta = \dfrac{c}{W}$. In the triangle with sides b and d and hypotenuse L, $\sin\theta = \dfrac{d}{L}$ and $\cos\theta = \dfrac{b}{L}$. Thus, $a = W\sin\theta$, $c = W\cos\theta$, $d = L\sin\theta$, and $b = L\cos\theta$, so the area of the circumscribed rectangle is

$$A(\theta) = (a+b)(c+d) = (W\sin\theta + L\cos\theta)(W\cos\theta + L\sin\theta)$$

$$= W^2\sin\theta\cos\theta + WL\sin^2\theta + LW\cos^2\theta + L^2\sin\theta\cos\theta$$

$$= LW\sin^2\theta + LW\cos^2\theta + (L^2+W^2)\sin\theta\cos\theta$$

$$= LW(\sin^2\theta + \cos^2\theta) + (L^2+W^2)\cdot\tfrac{1}{2}\cdot 2\sin\theta\cos\theta = LW + \tfrac{1}{2}(L^2+W^2)\sin 2\theta, \ 0 \le \theta \le \tfrac{\pi}{2}$$

This expression shows, without calculus, that the maximum value of $A(\theta)$ occurs when $\sin 2\theta = 1 \quad \Leftrightarrow \quad 2\theta = \frac{\pi}{2} \quad \Rightarrow$ $\theta = \frac{\pi}{4}$. So the maximum area is $A\left(\frac{\pi}{4}\right) = LW + \frac{1}{2}(L^2+W^2) = \frac{1}{2}(L^2+2LW+W^2) = \frac{1}{2}(L+W)^2$.

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86. (a) Let D be the point such that $a = |AD|$. From the text figure, $\sin\theta = \dfrac{b}{|BC|} \quad \Rightarrow \quad |BC| = b\csc\theta$ and

$\cos\theta = \dfrac{|BD|}{|BC|} = \dfrac{a-|AB|}{|BC|} \quad \Rightarrow \quad |BC| = (a-|AB|)\sec\theta$. Eliminating $|BC|$ gives

$(a-|AB|)\sec\theta = b\csc\theta \quad \Rightarrow \quad b\cot\theta = a-|AB| \quad \Rightarrow \quad |AB| = a - b\cot\theta$. The total resistance is

$R(\theta) = C\dfrac{|AB|}{r_1^4} + C\dfrac{|BC|}{r_2^4} = C\left(\dfrac{a-b\cot\theta}{r_1^4} + \dfrac{b\csc\theta}{r_2^4}\right)$.

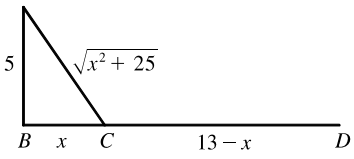
(b) $R'(\theta) = C\left(\dfrac{b\csc^2\theta}{r_1^4} - \dfrac{b\csc\theta\,\cot\theta}{r_2^4}\right) = bC\csc\theta\left(\dfrac{\csc\theta}{r_1^4} - \dfrac{\cot\theta}{r_2^4}\right)$.

$R'(\theta) = 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} = \dfrac{\cot\theta}{r_2^4} \quad \Leftrightarrow \quad \dfrac{r_2^4}{r_1^4} = \dfrac{\cot\theta}{\csc\theta} = \cos\theta.$

$R'(\theta) > 0 \quad \Leftrightarrow \quad \dfrac{\csc\theta}{r_1^4} > \dfrac{\cot\theta}{r_2^4} \quad \Rightarrow \quad \cos\theta < \dfrac{r_2^4}{r_1^4}$ and $R'(\theta) < 0$ when $\cos\theta > \dfrac{r_2^4}{r_1^4}$, so there is an absolute minimum when $\cos\theta = r_2^4/r_1^4$.

(c) When $r_2 = \frac{2}{3}r_1$, we have $\cos\theta = \left(\frac{2}{3}\right)^4$, so $\theta = \cos^{-1}\left(\frac{2}{3}\right)^4 \approx 79°$.

87. (a)



If k = energy/km over land, then energy/km over water $= 1.4k$.

So the total energy is $E = 1.4k\sqrt{25+x^2} + k(13-x),\ 0 \le x \le 13$,

and so $\dfrac{dE}{dx} = \dfrac{1.4kx}{(25+x^2)^{1/2}} - k$.

Set $\dfrac{dE}{dx} = 0$: $1.4kx = k(25+x^2)^{1/2} \quad \Rightarrow \quad 1.96x^2 = x^2+25 \quad \Rightarrow \quad 0.96x^2 = 25 \quad \Rightarrow \quad x = \frac{5}{\sqrt{0.96}} \approx 5.1$.

Testing against the value of E at the endpoints: $E(0) = 1.4k(5) + 13k = 20k$, $E(5.1) \approx 17.9k$, $E(13) \approx 19.5k$.

Thus, to minimize energy, the bird should fly to a point about 5.1 km from B.

(b) If W/L is large, the bird would fly to a point C that is closer to B than to D to minimize the energy used flying over water.

If W/L is small, the bird would fly to a point C that is closer to D than to B to minimize the distance of the flight.

$E = W\sqrt{25+x^2} + L(13-x) \quad \Rightarrow \quad \dfrac{dE}{dx} = \dfrac{Wx}{\sqrt{25+x^2}} - L = 0$ when $\dfrac{W}{L} = \dfrac{\sqrt{25+x^2}}{x}$. By the same sort of argument as in part (a), this ratio will give the minimal expenditure of energy if the bird heads for the point x km from B.

(c) For flight direct to D, $x = 13$, so from part (b), $W/L = \dfrac{\sqrt{25+13^2}}{13} \approx 1.07$. There is no value of W/L for which the bird should fly directly to B. But note that $\lim\limits_{x\to 0^+}(W/L) = \infty$, so if the point at which E is a minimum is close to B, then W/L is large.

(d) Assuming that the birds instinctively choose the path that minimizes the energy expenditure, we can use the equation for $dE/dx = 0$ from part (a) with $1.4k = c$, $x = 4$, and $k = 1$: $c(4) = 1\cdot(25+4^2)^{1/2} \quad \Rightarrow \quad c = \sqrt{41}/4 \approx 1.6$.

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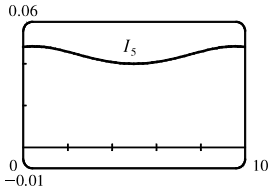
88. (a) $I(x) \propto \dfrac{\text{strength of source}}{(\text{distance from source})^2}$. Adding the intensities from the left and right lightbulbs,

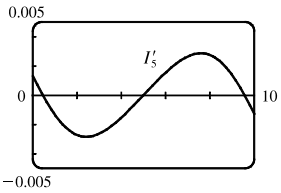
$$I(x) = \frac{k}{x^2+d^2} + \frac{k}{(10-x)^2+d^2} = \frac{k}{x^2+d^2} + \frac{k}{x^2-20x+100+d^2}.$$

(b) The magnitude of the constant k won't affect the location of the point of maximum intensity, so for convenience we take $k = 1$. $I'(x) = -\dfrac{2x}{(x^2+d^2)^2} - \dfrac{2(x-10)}{(x^2-20x+100+d^2)^2}$.

Substituting $d = 5$ into the equations for $I(x)$and $I'(x)$, we get

$$I_5(x) = \frac{1}{x^2+25} + \frac{1}{x^2-20x+125} \quad \text{and} \quad I_5'(x) = -\frac{2x}{(x^2+25)^2} - \frac{2(x-10)}{(x^2-20x+125)^2}$$

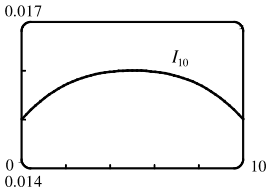


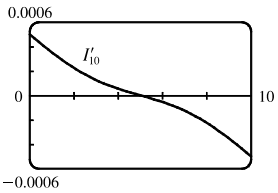


From the graphs, it appears that $I_5(x)$ has a minimum at $x = 5$ m.

(c) Substituting $d = 10$ into the equations for $I(x)$ and $I'(x)$ gives

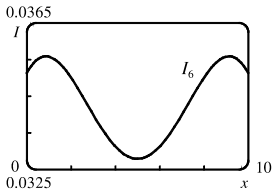
$$I_{10}(x) = \frac{1}{x^2+100} + \frac{1}{x^2-20x+200} \quad \text{and} \quad I_{10}'(x) = -\frac{2x}{(x^2+100)^2} - \frac{2\,(x-10)}{(x^2-20x+200)^2}$$

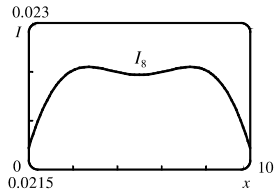


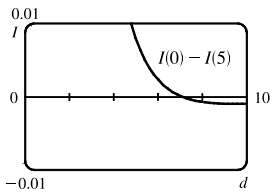


From the graphs, it seems that for $d = 10$, the intensity is minimized at the endpoints, that is, $x = 0$ and $x = 10$. The midpoint is now the most brightly lit point!

(d) From the first figures in parts (b) and (c), we see that the minimal illumination changes from the midpoint ($x = 5$ with $d = 5$) to the endpoints ($x = 0$ and $x = 10$ with $d = 10$).







So we try $d = 6$ (see the first figure) and we see that the minimum value still occurs at $x = 5$. Next, we let $d = 8$ (see the second figure) and we see that the minimum value occurs at the endpoints. It appears that for some value of d between 6 and 8, we must have minima at both the midpoint and the endpoints, that is, $I(5)$ must equal $I(0)$. To find this value of d, we solve $I(0) = I(5)$ (with $k = 1$):

$$\frac{1}{d^2} + \frac{1}{100+d^2} = \frac{1}{25+d^2} + \frac{1}{25+d^2} = \frac{2}{25+d^2} \quad \Rightarrow \quad (25+d^2)(100+d^2) + d^2(25+d^2) = 2d^2(100+d^2) \quad \Rightarrow$$

$2500 + 125d^2 + d^4 + 25d^2 + d^4 = 200d^2 + 2d^4 \quad \Rightarrow \quad 2500 = 50d^2 \quad \Rightarrow \quad d^2 = 50 \quad \Rightarrow \quad d = 5\sqrt{2} \approx 7.071$

[for $0 \le d \le 10$]. The third figure, a graph of $I(0) - I(5)$ with d independent, confirms that $I(0) - I(5) = 0$, that is,

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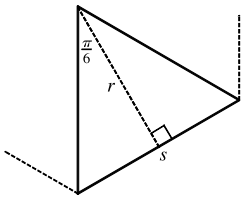
$I(0) = I(5)$, when $d = 5\sqrt{2}$. Thus, the point of minimal illumination changes abruptly from the midpoint to the endpoints when $d = 5\sqrt{2}$.

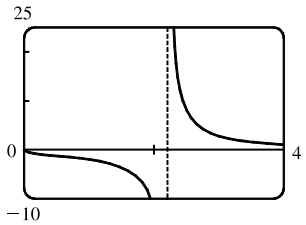
APPLIED PROJECT The Shape of a Can

1. In this case, the amount of metal used in the making of each top or bottom is $(2r)^2 = 4r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2(4r^2)$. But $V = \pi r^2 h \quad \Leftrightarrow \quad h = V/\pi r^2$. Substituting this expression for h in A gives $A = 2V/r + 8r^2$. Differentiating A with respect to r, we get $dA/dr = -2V/r^2 + 16r = 0 \quad \Rightarrow$ $16r^3 = 2V = 2\pi r^2 h \quad \Leftrightarrow \quad \dfrac{h}{r} = \dfrac{8}{\pi} \approx 2.55$. This gives a minimum because $\dfrac{d^2A}{dr^2} = 16 + \dfrac{4V}{r^3} > 0$.

2. 

We need to find the area of metal used up by each end, that is, the area of each hexagon. We subdivide the hexagon into six congruent triangles, each sharing one side (s in the diagram) with the hexagon. We calculate the length of $s = 2r\tan\frac{\pi}{6} = \frac{2}{\sqrt{3}}r$, so the area of each triangle is $\frac{1}{2}sr = \frac{1}{\sqrt{3}}r^2$, and the total area of the hexagon is $6 \cdot \frac{1}{\sqrt{3}}r^2 = 2\sqrt{3}\,r^2$. So the quantity we want to minimize is $A = 2\pi rh + 2 \cdot 2\sqrt{3}\,r^2$. Substituting for h as in Problem 1 and differentiating, we get $\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\sqrt{3}\,r$. Setting this equal to 0, we get $8\sqrt{3}\,r^3 = 2V = 2\pi r^2 h \quad \Rightarrow \quad \dfrac{h}{r} = \dfrac{4\sqrt{3}}{\pi} \approx 2.21$. Again this minimizes A because $\dfrac{d^2A}{dr^2} = 8\sqrt{3} + \dfrac{4V}{r^3} > 0$.

3. Let $C = 4\sqrt{3}\,r^2 + 2\pi rh + k\,(4\pi r + h) = 4\sqrt{3}\,r^2 + 2\pi r\left(\dfrac{V}{\pi r^2}\right) + k\left(4\pi r + \dfrac{V}{\pi r^2}\right)$. Then $\dfrac{dC}{dr} = 8\sqrt{3}\,r - \dfrac{2V}{r^2} + 4k\pi - \dfrac{2kV}{\pi r^3}$. Setting this equal to 0, dividing by 2 and substituting $\dfrac{V}{r^2} = \pi h$ and $\dfrac{V}{\pi r^3} = \dfrac{h}{r}$ in the second and fourth terms respectively, we get $0 = 4\sqrt{3}\,r - \pi h + 2k\pi - \dfrac{kh}{r} \quad \Leftrightarrow$ $k\left(2\pi - \dfrac{h}{r}\right) = \pi h - 4\sqrt{3}\,r \quad \Rightarrow \quad \dfrac{k}{r}\dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}} = 1$. We now multiply by $\dfrac{\sqrt[3]{V}}{k}$, noting that $\dfrac{\sqrt[3]{V}}{k}\dfrac{k}{r} = \sqrt[3]{\dfrac{V}{r^3}} = \sqrt[3]{\dfrac{\pi h}{r}}$, and get $\dfrac{\sqrt[3]{V}}{k} = \sqrt[3]{\dfrac{\pi h}{r}} \cdot \dfrac{2\pi - h/r}{\pi h/r - 4\sqrt{3}}$.

4. 

Let $\sqrt[3]{V}/k = T$ and $h/r = x$ so that $T(x) = \sqrt[3]{\pi x} \cdot \dfrac{2\pi - x}{\pi x - 4\sqrt{3}}$. We see from the graph of T that when the ratio $\sqrt[3]{V}/k$ is large; that is, either the volume of the can is large or the cost of joining (proportional to k) is small, the optimum value of h/r is about 2.21, but when $\sqrt[3]{V}/k$ is small, indicating small volume or expensive joining, the optimum value of h/r is larger. (The part of the graph for $\sqrt[3]{V}/k < 0$ has no physical meaning, but confirms the location of the asymptote.)

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5. Our conclusion is usually true in practice. But there are exceptions, such as cans of tuna, which may have to do with the shape of a reasonable slice of tuna. And for a comfortable grip on a soda or beer can, the geometry of the human hand is a restriction on the radius. Other possible considerations are packaging, transportation and stocking constraints, aesthetic appeal and other marketing concerns. Also, there may be better models than ours which prescribe a differently shaped can in special circumstances.

APPLIED PROJECT Planes and Birds: Minimizing Energy

1. $P(v) = Av^3 + \dfrac{BL^2}{v} \quad \Rightarrow \quad P'(v) = 3Av^2 - \dfrac{BL^2}{v^2}. \quad P'(v) = 0 \quad \Leftrightarrow \quad 3Av^2 = \dfrac{BL^2}{v^2} \quad \Leftrightarrow \quad v^4 = \dfrac{BL^2}{3A} \quad \Rightarrow$

$v = \sqrt[4]{\dfrac{BL^2}{3A}}. \quad P''(v) = 6Av + \dfrac{2BL^2}{v^3} > 0$, so the speed that minimizes the required power is $v_P = \left(\dfrac{BL^2}{3A}\right)^{1/4}$.

2. $E(v) = \dfrac{P(v)}{v} = Av^2 + \dfrac{BL^2}{v^2} \quad \Rightarrow \quad E'(v) = 2Av - \dfrac{2BL^2}{v^3}. \quad E'(v) = 0 \quad \Leftrightarrow \quad 2Av = \dfrac{2BL^2}{v^3} \quad \Leftrightarrow \quad v^4 = \dfrac{BL^2}{A} \quad \Rightarrow$

$v = \sqrt[4]{\dfrac{BL^2}{A}}. \quad E''(v) = 2A + \dfrac{6BL^2}{v^4} > 0$, so the speed that minimizes the energy needed to propel the plane is

$v_E = \left(\dfrac{BL^2}{A}\right)^{1/4}$.

3. $\dfrac{v_E}{v_P} = \dfrac{\left(\dfrac{BL^2}{A}\right)^{1/4}}{\left(\dfrac{BL^2}{3A}\right)^{1/4}} = \left(\dfrac{\dfrac{BL^2}{A}}{\dfrac{BL^2}{3A}}\right)^{1/4} = 3^{1/4} \approx 1.316$. Thus, $v_E \approx 1.316\, v_P$, so the speed for minimum energy is about 31.6% greater (faster) than the speed for minimum power.

4. Since x is the fraction of flying time spent in flapping mode, $1 - x$ is the fraction of time spent in folded mode. The average power $\overline{P}$ is the weighted average of P_{flap} and P_{fold}, so

$$\begin{aligned}\overline{P} &= xP_{\text{flap}} + (1-x)P_{\text{fold}} = x\left[(A_b + A_w)\, v^3 + \frac{B(mg/x)^2}{v}\right] + (1-x)A_b v^3 \\ &= xA_b v^3 + xA_w v^3 + x\,\frac{Bm^2 g^2}{x^2 v} + A_b v^3 - xA_b v^3 = A_b v^3 + xA_w v^3 + \frac{Bm^2 g^2}{xv}\end{aligned}$$

5. $\overline{P}(x) = A_b v^3 + xA_w v^3 + \dfrac{Bm^2 g^2}{xv} \quad \Rightarrow \quad \overline{P}'(x) = A_w v^3 - \dfrac{Bm^2 g^2}{x^2 v}. \quad \overline{P}'(x) = 0 \quad \Leftrightarrow \quad A_w v^3 = \dfrac{Bm^2 g^2}{x^2 v} \quad \Leftrightarrow$

$x^2 = \dfrac{Bm^2 g^2}{A_w v^4} \quad \Rightarrow \quad x = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$. Since $\overline{P}''(x) = \dfrac{2Bm^2 g^2}{x^3 v} > 0$, this critical number, call it $x_{\overline{P}}$, gives an absolute minimum for the average power. If the bird flies slowly, then v is smaller and $x_{\overline{P}}$ increases, and the bird spends a larger fraction of its flying time flapping. If the bird flies faster and faster, then v is larger and $x_{\overline{P}}$ decreases, and the bird spends a smaller fraction of its flying time flapping, while still minimizing average power.

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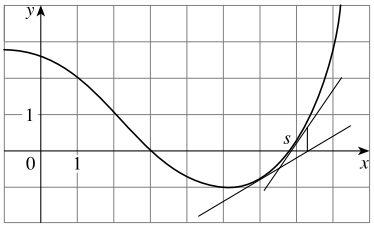
6. $\overline{E}(x) = \dfrac{\overline{P}(x)}{v} \;\Rightarrow\; \overline{E}'(x) = \dfrac{1}{v}\overline{P}'(x)$, so $\overline{E}'(x) = 0 \;\Leftrightarrow\; \overline{P}'(x) = 0$. The value of x that minimizes $\overline{E}$ is the same value of x that minimizes $\overline{P}$, namely $x_{\overline{P}} = \dfrac{mg}{v^2}\sqrt{\dfrac{B}{A_w}}$.

4.8 Newton's Method

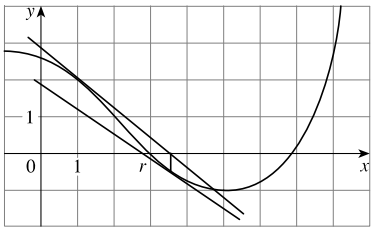
1. (a)



The tangent line at $x_1 = 6$ intersects the x-axis at $x \approx 7.3$, so $x_2 = 7.3$. The tangent line at $x = 7.3$ intersects the x-axis at $x \approx 6.8$, so $x_3 \approx 6.8$.

(b) $x_1 = 8$ would be a better first approximation because the tangent line at $x = 8$ intersects the x-axis closer to s than does the first approximation $x_1 = 6$.

2.



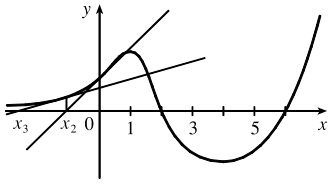
The tangent line at $x_1 = 1$ intersects the x-axis at $x \approx 3.5$, so $x_2 = 3.5$. The tangent line at $x = 3.5$ intersects the x-axis at $x \approx 2.8$, so $x_3 = 2.8$.

3. Since the tangent line $y = 9 - 2x$ is tangent to the curve $y = f(x)$ at the point $(2, 5)$, we have $x_1 = 2$, $f(x_1) = 5$, and $f'(x_1) = -2$ [the slope of the tangent line]. Thus, by Equation 2,

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2 - \frac{5}{-2} = \frac{9}{2}$$

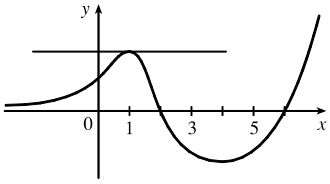
Note that geometrically $\frac{9}{2}$ represents the x-intercept of the tangent line $y = 9 - 2x$.

4. (a)



If $x_1 = 0$, then x_2 is negative, and x_3 is even more negative. The sequence of approximations does not converge, that is, Newton's method fails.

(b)



If $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

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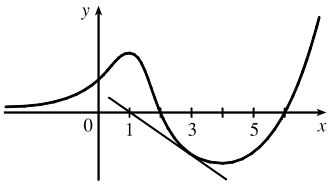
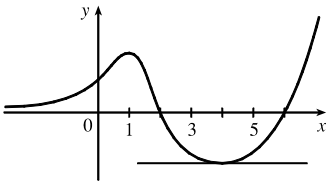
(c)



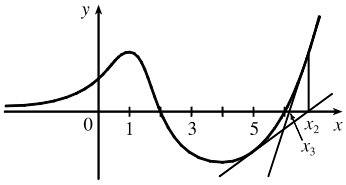
If $x_1 = 3$, then $x_2 = 1$ and we have the same situation as in part (b). Newton's method fails again.

(d)



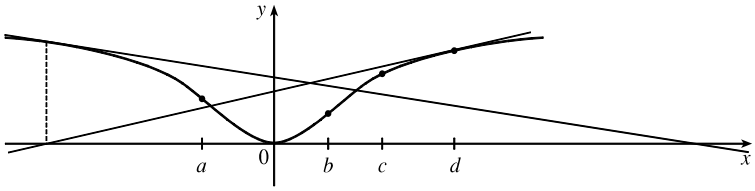
If $x_1 = 4$, the tangent line is horizontal and Newton's method fails.

(e)



If $x_1 = 5$, then x_2 is greater than 6, x_3 gets closer to 6, and the sequence of approximations converges to 6. Newton's method succeeds!

5. The initial approximations $x_1 = a, b$, and c will work, resulting in a second approximation closer to the origin, and lead to the solution of the equation $f(x) = 0$, namely, $x = 0$. The initial approximation $x_1 = d$ will not work because it will result in successive approximations farther and farther from the origin.



6. $f(x) = 2x^3 - 3x^2 + 2 \;\Rightarrow\; f'(x) = 6x^2 - 6x$, so $x_{n+1} = x_n - \dfrac{2x_n^3 - 3x_n^2 + 2}{6x_n^2 - 6x_n}$. Now $x_1 = -1 \;\Rightarrow$

$$x_2 = -1 - \frac{2(-1)^3 - 3(-1)^2 + 2}{6(-1)^2 - 6(-1)} = -1 - \frac{-3}{12} = -\frac{3}{4} \;\Rightarrow$$

$$x_3 = -\frac{3}{4} - \frac{2\left(-\frac{3}{4}\right)^3 - 3\left(-\frac{3}{4}\right)^2 + 2}{6\left(-\frac{3}{4}\right)^2 - 6\left(-\frac{3}{4}\right)} = -\frac{3}{4} - \frac{-17/32}{63/8} = -\frac{43}{63} \approx -0.6825.$$

7. $f(x) = \dfrac{2}{x} - x^2 + 1 \;\Rightarrow\; f'(x) = -\dfrac{2}{x^2} - 2x$, so $x_{n+1} = x_n - \dfrac{2/x_n - x_n^2 + 1}{-2/x_n^2 - 2x_n}$. Now $x_1 = 2 \;\Rightarrow$

$$x_2 = 2 - \frac{1 - 4 + 1}{-1/2 - 4} = 2 - \frac{-2}{-9/2} = \frac{14}{9} \;\Rightarrow\; x_3 = \frac{14}{9} - \frac{2/(14/9) - (14/9)^2 + 1}{-2(14/9)^2 - 2(14/9)} \approx 1.5215.$$

8. Solving $x^5 = x^2 + 1$ is the same as solving $f(x) = x^5 - x^2 - 1 = 0$. $f'(x) = 5x^4 - 2x$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^2 - 1}{5x_n^4 - 2x_n}$.

Now, $x_1 = 1 \;\Rightarrow\; x_2 = 1 - \dfrac{1^5 - 1^2 - 1}{5(1)^4 - 2(1)} = 1 - \dfrac{-1}{3} = \dfrac{4}{3} \;\Rightarrow\; x_3 = \dfrac{4}{3} - \dfrac{(4/3)^5 - (4/3)^2 - 1}{5(4/3)^4 - 2(4/3)} \approx 1.2240.$

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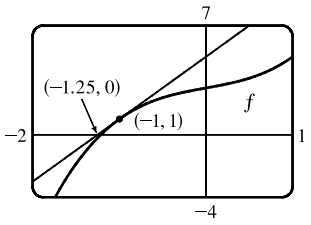
9. $f(x) = x^3 + x + 3 \quad \Rightarrow \quad f'(x) = 3x^2 + 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 + x_n + 3}{3x_n^2 + 1}$.

Now $x_1 = -1 \quad \Rightarrow$

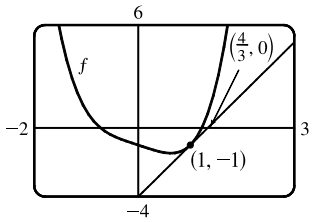
$$x_2 = -1 - \frac{(-1)^3 + (-1) + 3}{3(-1)^2 + 1} = -1 - \frac{-1 - 1 + 3}{3 + 1} = -1 - \frac{1}{4} = -1.25.$$

Newton's method follows the tangent line at $(-1, 1)$ down to its intersection with the x-axis at $(-1.25, 0)$, giving the second approximation $x_2 = -1.25$.



10. $f(x) = x^4 - x - 1 \quad \Rightarrow \quad f'(x) = 4x^3 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^4 - x_n - 1}{4x_n^3 - 1}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1 - \dfrac{1^4 - 1 - 1}{4 \cdot 1^3 - 1} = 1 - \dfrac{-1}{3} = \dfrac{4}{3}$. Newton's method follows the tangent line at $(1, -1)$ up to its intersection with the x-axis at $\left(\frac{4}{3}, 0\right)$, giving the second approximation $x_2 = \frac{4}{3}$.



11. To approximate $x = \sqrt[4]{75}$ (so that $x^4 = 75$), we can take $f(x) = x^4 - 75$. So $f'(x) = 4x^3$, and thus, $x_{n+1} = x_n - \dfrac{x_n^4 - 75}{4x_n^3}$. Since $\sqrt[4]{81} = 3$ and 81 is reasonably close to 75, we'll use $x_1 = 3$. We need to find approximations until they agree to eight decimal places. $x_1 = 3 \quad \Rightarrow \quad x_2 = 2.9\overline{4}$, $x_3 \approx 2.94283228$, $x_4 \approx 2.94283096 \approx x_5$. So $\sqrt[4]{75} \approx 2.94283096$, to eight decimal places.

To use Newton's method on a calculator, assign f to Y_1 and f' to Y_2. Then store x_1 in X and enter $\mathrm{X} - \mathrm{Y}_1/\mathrm{Y}_2 \rightarrow \mathrm{X}$ to get x_2 and further approximations (repeatedly press ENTER).

12. $f(x) = x^8 - 500 \quad \Rightarrow \quad f'(x) = 8x^7$, so $x_{n+1} = x_n - \dfrac{x_n^8 - 500}{8x_n^7}$. Since $\sqrt[8]{256} = 2$ and 256 is reasonably close to 500, we'll use $x_1 = 2$. We need to find approximations until they agree to eight decimal places. $x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.23828125$, $x_3 \approx 2.18055972$, $x_4 \approx 2.17461675$, $x_5 \approx 2.17455928 \approx x_6$. So $\sqrt[8]{500} \approx 2.17455928$, to eight decimal places.

13. (a) Let $f(x) = 3x^4 - 8x^3 + 2$. The polynomial f is continuous on $[2, 3]$, $f(2) = -14 < 0$, and $f(3) = 29 > 0$, so by the Intermediate Value Theorem, there is a number c in $(2, 3)$ such that $f(c) = 0$. In other words, the equation $3x^4 - 8x^3 + 2 = 0$ has a solution in $[2, 3]$.

(b) $f'(x) = 12x^3 - 24x^2 \quad \Rightarrow \quad x_{n+1} = x_n - \dfrac{3x_n^4 - 8x_n^3 + 2}{12x_n^3 - 24x_n^2}$. Taking $x_1 = 2.5$, we get $x_2 = 2.655$, $x_3 \approx 2.630725$, $x_4 \approx 2.630021$, $x_5 \approx 2.630020 \approx x_6$. To six decimal places, the solution is 2.630020. Note that taking $x_1 = 2$ is not allowed since $f'(2) = 0$.

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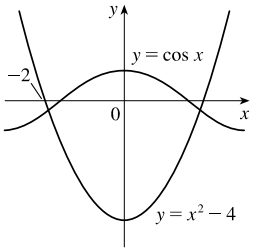
14. (a) Let $f(x) = -2x^5 + 9x^4 - 7x^3 - 11x$. The polynomial f is continuous on $[3, 4]$, $f(3) = 21 > 0$, and $f(4) = -236 < 0$, so by the Intermediate Value Theorem, there is a number c in $(3, 4)$ such that $f(c) = 0$. In other words, the equation $-2x^5 + 9x^4 - 7x^3 - 11x = 0$ has a solution in $[3, 4]$.

(b) $f'(x) = -10x^4 + 36x^3 - 21x^2 - 11$. $x_{n+1} = x_n - \dfrac{-2x_n^5 + 9x_n^4 - 7x_n^3 - 11x_n}{-10x_n^4 + 36x_n^3 - 21x_n^2 - 11}$. Taking $x_1 = 3.5$, we get $x_2 \approx 3.329174$, $x_3 = 3.278706$, $x_4 \approx 3.274501$, and $x_5 \approx 3.274473 \approx x_6$. To six decimal places, the solution is 3.274473.

15.

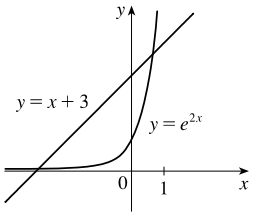


From the graph, we see that the negative solution of $\cos x = x^2 - 4$ is near $x = -2$. Solving $\cos x = x^2 - 4$ is the same as solving $f(x) = \cos x - x^2 + 4 = 0$.

$f'(x) = -\sin x - 2x$, so $x_{n+1} = x_n - \dfrac{\cos x_n - x_n^2 + 4}{-\sin x_n - 2x_n}$. $x_1 = -2 \Rightarrow$

$x_2 \approx -1.915233$, $x_3 \approx -1.914021 \approx x_4$. Thus, the negative solution is -1.914021, to six decimal places.

16.

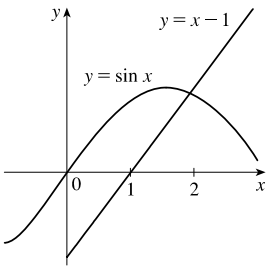


From the graph, we see that the positive solution of $e^{2x} = x + 3$ is near $x = 1$. Solving $e^{2x} = x + 3$ is the same as solving $f(x) = e^{2x} - x - 3 = 0$.

$f'(x) = 2e^{2x} - 1$, so $x_{n+1} = x_n - \dfrac{e^{2x_n} - x_n - 3}{2e^{2x_n} - 1}$. $x_1 = 1 \Rightarrow$

$x_2 \approx 0.754026$, $x_3 \approx 0.658965$, $x_4 \approx 0.647110$, $x_5 \approx 0.646945 \approx x_6$. Thus, the positive solution is 0.646945, to six decimal places.

17.

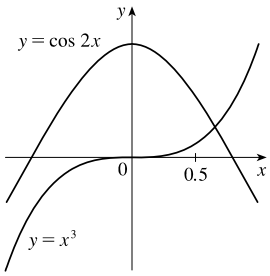


From the graph, we see that there appears to be a point of intersection near $x = 2$. Solving $\sin x = x - 1$ is the same as solving $f(x) = \sin x - x + 1 = 0$.

$f'(x) = \cos x - 1$, so $x_{n+1} = x_n - \dfrac{\sin x_n - x_n + 1}{\cos x_n - 1}$. $x_1 = 2 \Rightarrow$

$x_2 \approx 1.935951$, $x_3 \approx 1.934564$, $x_4 \approx 1.934563 \approx x_5$. Thus, the solution is 1.934563, to six decimal places.

18.



From the graph, we see that there appears to be a point of intersection near $x = 0.5$. Solving $\cos 2x = x^3$ is the same as solving $f(x) = \cos 2x - x^3 = 0$.

$f'(x) = -2\sin 2x - 3x^2$, so $x_{n+1} = x_n - \dfrac{\cos 2x_n - x_n^3}{-2\sin 2x_n - 3x_n^2}$. $x_1 = 0.5 \Rightarrow$

$x_2 \approx 0.670700$, $x_3 \approx 0.648160$, $x_4 \approx 0.647766$, $x_5 \approx 0.647765 \approx x_6$. Thus, the solution 0.647765, to six decimal places.

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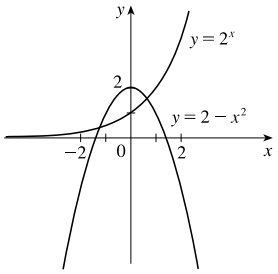
19.



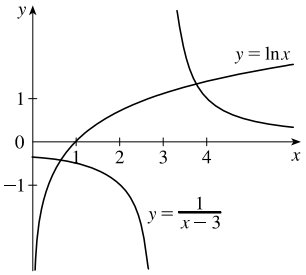
From the figure, we see that the graphs intersect between -2 and -1 and between 0 and 1. Solving $2^x = 2 - x^2$ is the same as solving $f(x) = 2^x - 2 + x^2 = 0$. $f'(x) = 2^x \ln 2 + 2x$, so

$$x_{n+1} = x_n - \frac{2^{x_n} - 2 + x_n^2}{2^{x_n} \ln 2 + 2x_n}.$$

$x_1 = -1$	$x_1 = 1$
$x_2 \approx -1.302402$	$x_2 \approx 0.704692$
$x_3 \approx -1.258636$	$x_3 \approx 0.654915$
$x_4 \approx -1.257692$	$x_4 \approx 0.653484$
$x_5 \approx -1.257691 \approx x_6$	$x_5 \approx 0.653483 \approx x_6$

To six decimal places, the solutions of the equation are -1.257691 and 0.653483.

20.

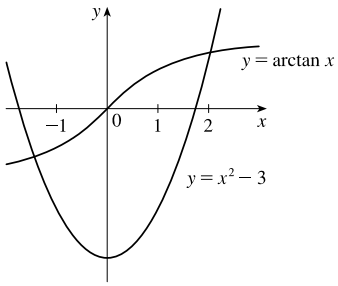


From the figure, we see that the graphs intersect between 0 and 1 and between 3 and 4. Solving $\ln x = \dfrac{1}{x-3}$ is the same as solving $f(x) = \ln x - \dfrac{1}{x-3} = 0$. $f'(x) = \dfrac{1}{x} + \dfrac{1}{(x-3)^2}$, so $x_{n+1} = x_n - \dfrac{\ln x_n - 1/(x_n - 3)}{(1/x_n) + 1/(x_n - 3)^2}$.

$x_1 = 1$	$x_1 = 4$
$x_2 \approx 0.6$	$x_2 \approx 3.690965$
$x_3 \approx 0.651166$	$x_3 \approx 3.750726$
$x_4 \approx 0.653057$	$x_4 \approx 3.755672$
$x_5 \approx 0.653060 \approx x_6$	$x_5 \approx 3.755701 \approx x_6$

To six decimal places, the solutions of the equation are 0.653060 and 3.755701.

21.



From the figure, we see that there appear to be points of intersection near $x = -1$ and $x = 2$. Solving $\arctan x = x^2 - 3$ is the same as solving $f(x) = \arctan x - x^2 + 3 = 0$. $f'(x) = \dfrac{1}{1+x^2} - 2x$, so

$$x_{n+1} = x_n - \frac{\arctan x_n - x_n^2 + 3}{\dfrac{1}{1+x_n^2} - 2x_n}.$$

$x_1 = -1$	$x_1 = 2$
$x_2 \approx -1.485841$	$x_2 \approx 2.028197$
$x_3 \approx -1.429153$	$x_3 \approx 2.027975 \approx x_4$
$x_4 \approx -1.428293 \approx x_5$	

To six decimal places, the solutions of the equation are -1.428293 and 2.027975.

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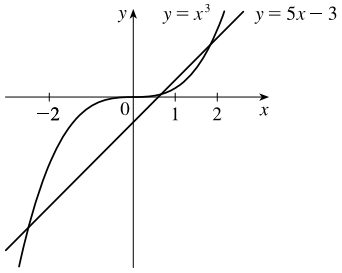
22.



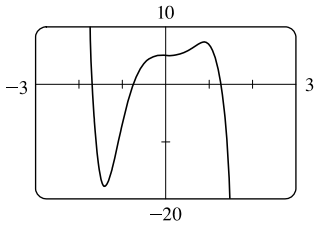
From the figure, we see that there appear to be points of intersection near $x = -2$, $x = 1$, and $x = 2$. Solving $x^3 = 5x - 3$ is the same as solving $f(x) = x^3 - 5x + 3 = 0$. $f'(x) = 3x^2 - 5$, so

$$x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5}.$$

$x_1 = -2$	$x_1 = 1$	$x_1 = 2$
$x_2 \approx -2.714286$	$x_2 = 0.5$	$x_2 \approx 1.857143$
$x_3 \approx -2.513979$	$x_3 \approx 0.647059$	$x_3 \approx 1.834787$
$x_4 \approx -2.491151$	$x_4 \approx 0.656573$	$x_4 \approx 1.834244$
$x_5 \approx -2.490864 \approx x_6$	$x_5 \approx 0.656620 \approx x_6$	$x_5 \approx 1.834243 \approx x_6$

To six decimal places, the solutions of the equation are -2.490864, 0.656620, and 1.834243.

23.



$f(x) = -2x^7 - 5x^4 + 9x^3 + 5 \quad \Rightarrow \quad f'(x) = -14x^6 - 20x^3 + 27x^2 \quad \Rightarrow$

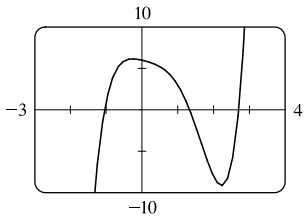
$$x_{n+1} = x_n - \frac{-2x_n^7 - 5x_n^4 + 9x_n^3 + 5}{-14x_n^6 - 20x_n^3 + 27x_n^2}.$$

From the graph of f, there appear to be solutions near -1.7, -0.7, and 1.3.

$x_1 = -1.7$	$x_1 = -0.7$	$x_1 = 1.3$
$x_2 = -1.693255$	$x_2 \approx -0.74756345$	$x_2 = 1.268776$
$x_3 \approx -1.69312035$	$x_3 \approx -0.74467752$	$x_3 \approx 1.26589387$
$x_4 \approx -1.69312029 \approx x_5$	$x_4 \approx -0.74466668 \approx x_5$	$x_4 \approx 1.26587094 \approx x_5$

To eight decimal places, the solutions of the equation are -1.69312029, -0.74466668, and 1.26587094.

24.



$f(x) = x^5 - 3x^4 + x^3 - x^2 - x + 6 \quad \Rightarrow$

$f'(x) = 5x^4 - 12x^3 + 3x^2 - 2x - 1 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{x_n^5 - 3x_n^4 + x_n^3 - x_n^2 - x_n + 6}{5x_n^4 - 12x_n^3 + 3x_n^2 - 2x_n - 1}$. From the graph of f, there appear to be solutions near -1, 1.3, and 2.7.

$x_1 = -1$	$x_1 = 1.3$	$x_1 = 2.7$
$x_2 \approx -1.04761905$	$x_2 \approx 1.33313045$	$x_2 \approx 2.70556135$
$x_3 \approx -1.04451724$	$x_3 \approx 1.33258330$	$x_3 \approx 2.70551210$
$x_4 \approx -1.04450307 \approx x_5$	$x_4 \approx 1.33258316 \approx x_5$	$x_4 \approx 2.70551209 \approx x_5$

To eight decimal places, the solutions of the equation are -1.04450307, 1.33258316, and 2.70551209.

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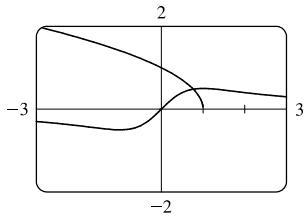
25.



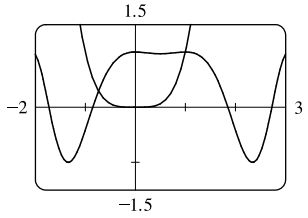
Solving $\dfrac{x}{x^2+1} = \sqrt{1-x}$ is the same as solving

$f(x) = \dfrac{x}{x^2+1} - \sqrt{1-x} = 0$. $f'(x) = \dfrac{1-x^2}{(x^2+1)^2} + \dfrac{1}{2\sqrt{1-x}} \quad \Rightarrow$

$$x_{n+1} = x_n - \frac{\dfrac{x_n}{x_n^2+1} - \sqrt{1-x_n}}{\dfrac{1-x_n^2}{(x_n^2+1)^2} + \dfrac{1}{2\sqrt{1-x_n}}}.$$

From the graph, we see that the curves intersect at about 0.8. $x_1 = 0.8 \quad \Rightarrow \quad x_2 \approx 0.76757581$, $x_3 \approx 0.76682610$, $x_4 \approx 0.76682579 \approx x_5$. To eight decimal places, the solution of the equation is 0.76682579.

26.

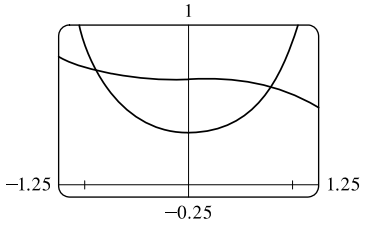


Solving $\cos(x^2 - x) = x^4$ is the same as solving

$f(x) = \cos(x^2 - x) - x^4 = 0$. $f'(x) = -(2x-1)\sin(x^2-x) - 4x^3 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{\cos(x_n^2 - x_n) - x_n^4}{-(2x_n - 1)\sin(x_n^2 - x_n) - 4x_n^3}$. From the equations $y = \cos(x^2 - x)$ and $y = x^4$ and the graph, we deduce that one solution of the equation $\cos(x^2 - x) = x^4$ is $x = 1$. We also see that the graphs intersect at approximately $x = -0.7$. $x_1 = -0.7 \quad \Rightarrow \quad x_2 \approx -0.73654354$, $x_3 \approx -0.73486274$, $x_4 \approx -0.73485910 \approx x_5$. To eight decimal places, one solution of the equation is -0.73485910; the other solution is 1.

27.



From the graph, we see that the curves intersect at approximately $x = -0.8$ and $x = 0.8$. Solving $\sqrt{4-x^3} = e^{x^2}$ is the same as solving

$f(x) = \sqrt{4-x^3} - e^{x^2} = 0$.

$$f'(x) = \frac{1}{2\sqrt{4-x^3}} \cdot (-3x^2) - e^{x^2} \cdot 2x = -\frac{3x^2}{2\sqrt{4-x^3}} - 2xe^{x^2} \quad \Rightarrow \quad x_{n+1} = x_n - \frac{\sqrt{4-x_n^3} - e^{x_n^2}}{-\dfrac{3x_n^2}{2\sqrt{4-x_n^3}} - 2x_n e^{x_n^2}}.$$

$x_1 = -0.8$	$x_1 = 0.8$
$x_2 = -0.88815983$	$x_2 \approx 0.79186616$
$x_3 \approx -0.87842996$	$x_3 \approx 0.79177078$
$x_4 \approx -0.87828296$	$x_4 \approx 0.79177077 \approx x_5$
$x_5 \approx -0.87828292 \approx x_6$	

To eight decimal places, the solutions of the equation are -0.87828292 and 0.79177077.

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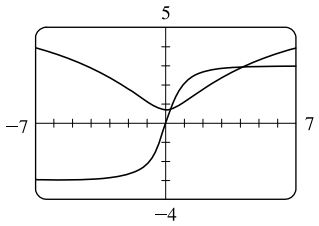
28.



Solving $\ln(x^2+2) = \dfrac{3x}{\sqrt{x^2+1}}$ is the same as solving

$$f(x) = \ln(x^2+2) - \frac{3x}{\sqrt{x^2+1}} = 0.$$

$$f'(x) = \frac{2x}{x^2+2} - \frac{(x^2+1)^{1/2}(3) - (3x)\frac{1}{2}(x^2+1)^{-1/2}(2x)}{[(x^2+1)^{1/2}]^2}$$

$$= \frac{2x}{x^2+2} - \frac{(x^2+1)^{-1/2}[3(x^2+1) - 3x^2]}{(x^2+1)^1}$$

$$= \frac{2x}{x^2+2} - \frac{3}{(x^2+1)^{3/2}} \quad \Rightarrow \quad x_{n+1} = x_n - \frac{\ln(x_n^2+2) - \dfrac{3x_n}{\sqrt{x_n^2+1}}}{\dfrac{2x_n}{x_n^2+2} - \dfrac{3}{(x_n^2+1)^{3/2}}}.$$

From the figure, we see that the graphs intersect at approximately $x = 0.2$ and $x = 4$.

$x_1 = 0.2$	$x_1 = 4$
$x_2 \approx 0.24733161$	$x_2 \approx 4.04993412$
$x_3 \approx 0.24852333$	$x_3 \approx 4.05010983$
$x_4 \approx 0.24852414 \approx x_5$	$x_4 \approx 4.05010984 \approx x_5$

To eight decimal places, the solutions of the equation are 0.24852414 and 4.05010984.

29. (a) $f(x) = x^2 - a \quad \Rightarrow \quad f'(x) = 2x$, so Newton's method gives

$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n} = x_n - \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}x_n + \frac{a}{2x_n} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Using (a) with $a = 1000$ and $x_1 = \sqrt{900} = 30$, we get $x_2 \approx 31.666667$, $x_3 \approx 31.622807$, and $x_4 \approx 31.622777 \approx x_5$. So $\sqrt{1000} \approx 31.622777$.

30. (a) $f(x) = \dfrac{1}{x} - a \quad \Rightarrow \quad f'(x) = -\dfrac{1}{x^2}$, so $x_{n+1} = x_n - \dfrac{1/x_n - a}{-1/x_n^2} = x_n + x_n - ax_n^2 = 2x_n - ax_n^2$.

(b) Using (a) with $a = 1.6894$ and $x_1 = \frac{1}{2} = 0.5$, we get $x_2 = 0.5754$, $x_3 \approx 0.588485$, and $x_4 \approx 0.588789 \approx x_5$. So $1/1.6984 \approx 0.588789$.

31. $f(x) = x^3 - 3x + 6 \quad \Rightarrow \quad f'(x) = 3x^2 - 3$. If $x_1 = 1$, then $f'(x_1) = 0$ and the tangent line used for approximating x_2 is horizontal. Attempting to find x_2 results in trying to divide by zero.

32. $x^3 - x = 1 \quad \Leftrightarrow \quad x^3 - x - 1 = 0$. $f(x) = x^3 - x - 1 \quad \Rightarrow \quad f'(x) = 3x^2 - 1$, so $x_{n+1} = x_n - \dfrac{x_n^3 - x_n - 1}{3x_n^2 - 1}$.

(a) $x_1 = 1$, $x_2 = 1.5$, $x_3 \approx 1.347826$, $x_4 \approx 1.325200$, $x_5 \approx 1.324718 \approx x_6$

(b) $x_1 = 0.6$, $x_2 = 17.9$, $x_3 \approx 11.946802$, $x_4 \approx 7.985520$, $x_5 \approx 5.356909$, $x_6 \approx 3.624996$, $x_7 \approx 2.505589$, $x_8 \approx 1.820129$, $x_9 \approx 1.461044$, $x_{10} \approx 1.339323$, $x_{11} \approx 1.324913$, $x_{12} \approx 1.324718 \approx x_{13}$

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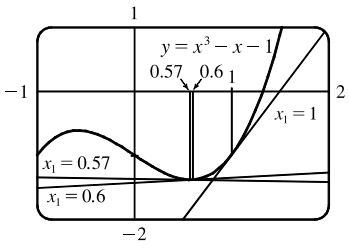
(c) $x_1 = 0.57$, $x_2 \approx -54.165455$, $x_3 \approx -36.114293$, $x_4 \approx -24.082094$, $x_5 \approx -16.063387$, $x_6 \approx -10.721483$, $x_7 \approx -7.165534$, $x_8 \approx -4.801704$, $x_9 \approx -3.233425$, $x_{10} \approx -2.193674$, $x_{11} \approx -1.496867$, $x_{12} \approx -0.997546$, $x_{13} \approx -0.496305$, $x_{14} \approx -2.894162$, $x_{15} \approx -1.967962$, $x_{16} \approx -1.341355$, $x_{17} \approx -0.870187$, $x_{18} \approx -0.249949$, $x_{19} \approx -1.192219$, $x_{20} \approx -0.731952$, $x_{21} \approx 0.355213$, $x_{22} \approx -1.753322$, $x_{23} \approx -1.189420$, $x_{24} \approx -0.729123$, $x_{25} \approx 0.377844$, $x_{26} \approx -1.937872$, $x_{27} \approx -1.320350$, $x_{28} \approx -0.851919$, $x_{29} \approx -0.200959$, $x_{30} \approx -1.119386$, $x_{31} \approx -0.654291$, $x_{32} \approx 1.547010$, $x_{33} \approx 1.360051$, $x_{34} \approx 1.325828$, $x_{35} \approx 1.324719$, $x_{36} \approx 1.324718 \approx x_{37}$.

(d)

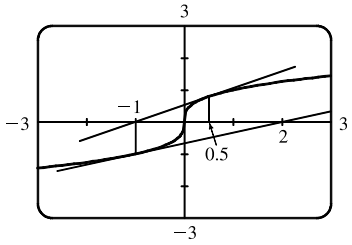


From the figure, we see that the tangent line corresponding to $x_1 = 1$ results in a sequence of approximations that converges quite quickly ($x_5 \approx x_6$). The tangent line corresponding to $x_1 = 0.6$ is close to being horizontal, so x_2 is quite far from the solution. But the sequence still converges — just a little more slowly ($x_{12} \approx x_{13}$). Lastly, the tangent line corresponding to $x_1 = 0.57$ is very nearly horizontal, x_2 is farther away from the solution, and the sequence takes more iterations to converge ($x_{36} \approx x_{37}$).

33. For $f(x) = x^{1/3}$, $f'(x) = \frac{1}{3}x^{-2/3}$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n.$$

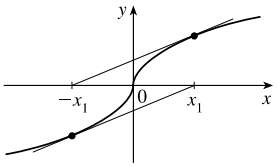
Therefore, each successive approximation becomes twice as large as the previous one in absolute value, so the sequence of approximations fails to converge to the solution, which is 0. In the figure, we have $x_1 = 0.5$, $x_2 = -2(0.5) = -1$, and $x_3 = -2(-1) = 2$.



34. According to Newton's Method, for $x_n > 0$,

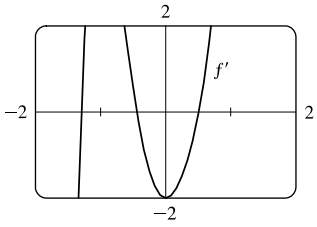
$$x_{n+1} = x_n - \frac{\sqrt{x_n}}{1/\left(2\sqrt{x_n}\right)} = x_n - 2x_n = -x_n \text{ and for } x_n < 0,$$

$$x_{n+1} = x_n - \frac{-\sqrt{-x_n}}{1/\left(2\sqrt{-x_n}\right)} = x_n - [-2(-x_n)] = -x_n.$$ So we can see that after choosing any value x_1 the subsequent values will alternate between $-x_1$ and x_1 and never approach the solution.



35. (a) $f(x) = x^6 - x^4 + 3x^3 - 2x \Rightarrow f'(x) = 6x^5 - 4x^3 + 9x^2 - 2 \Rightarrow f''(x) = 30x^4 - 12x^2 + 18x$. To find the critical numbers of f, we'll find the zeros of f'. From the graph of f', it appears there are zeros at approximately $x = -1.3$, -0.4, and 0.5. Try $x_1 = -1.3 \Rightarrow$

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} \approx -1.293344 \Rightarrow x_3 \approx -1.293227 \approx x_4.$$



Now try $x_1 = -0.4 \Rightarrow x_2 \approx -0.443755 \Rightarrow x_3 \approx -0.441735 \Rightarrow x_4 \approx -0.441731 \approx x_5$. Finally try

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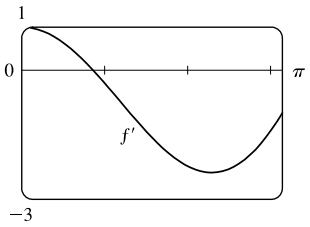
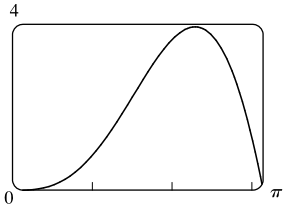
$x_1 = 0.5 \Rightarrow x_2 \approx 0.507937 \Rightarrow x_3 \approx 0.507854 \approx x_4$. Therefore, $x = -1.293227$, -0.441731, and 0.507854 are all the critical numbers correct to six decimal places.

(b) There are two critical numbers where f' changes from negative to positive, so f changes from decreasing to increasing. $f(-1.293227) \approx -2.0212$ and $f(0.507854) \approx -0.6721$, so -2.0212 is the absolute minimum value of f correct to four decimal places.

36. $f(x) = x\cos x \Rightarrow f'(x) = \cos x - x\sin x$. $f'(x)$ exists for all x, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for x_1 is $x_1 = 0.9$. Use $g(x) = \cos x - x\sin x$ and $g'(x) = -2\sin x - x\cos x$ to obtain $x_2 \approx 0.860781$, $x_3 \approx 0.860334 \approx x_4$. Now we have $f(0) = 0$, $f(\pi) = -\pi$, and $f(0.860334) \approx 0.561096$, so 0.561096 is the absolute maximum value of f correct to six decimal places.



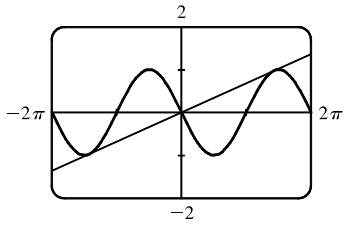
37.



$$y = x^2\sin x \Rightarrow y' = x^2\cos x + (\sin x)(2x) \Rightarrow$$
$$y'' = x^2(-\sin x) + (\cos x)(2x) + (\sin x)(2) + 2x\cos x$$
$$= -x^2\sin x + 4x\cos x + 2\sin x \Rightarrow$$
$$y''' = -x^2\cos x + (\sin x)(-2x) + 4x(-\sin x) + (\cos x)(4) + 2\cos x$$
$$= -x^2\cos x - 6x\sin x + 6\cos x.$$

From the graph of $y = x^2\sin x$, we see that $x = 1.5$ is a reasonable guess for the x-coordinate of the inflection point. Using Newton's method with $g(x) = y''$ and $g'(x) = y'''$, we get $x_1 = 1.5 \Rightarrow x_2 \approx 1.520092$, $x_3 \approx 1.519855 \approx x_4$. The inflection point is about $(1.519855, 2.306964)$.

38.



$f(x) = -\sin x \Rightarrow f'(x) = -\cos x$. At $x = a$, the slope of the tangent line is $f'(a) = -\cos a$. The line through the origin and $(a, f(a))$ is $y = \dfrac{-\sin a - 0}{a - 0}x$. If this line is to be tangent to f at $x = a$, then its slope must equal $f'(a)$. Thus, $\dfrac{-\sin a}{a} = -\cos a \Rightarrow \tan a = a$.

To solve this equation using Newton's method, let $g(x) = \tan x - x$, $g'(x) = \sec^2 x - 1$, and $x_{n+1} = x_n - \dfrac{\tan x_n - x_n}{\sec^2 x_n - 1}$ with $x_1 = 4.5$ (estimated from the figure). $x_2 \approx 4.493614$, $x_3 \approx 4.493410$, $x_4 \approx 4.493409 \approx x_5$. Thus, the slope of the line that has the largest slope is $f'(x_5) \approx 0.217234$.

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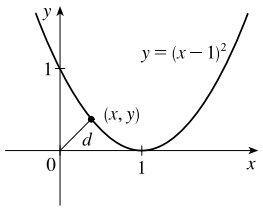
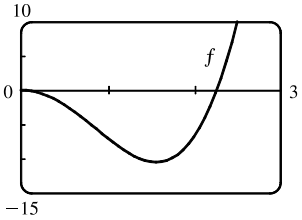
39. We need to minimize the distance from $(0,0)$ to an arbitrary point (x,y) on the curve $y=(x-1)^2$. $d=\sqrt{x^2+y^2}$ $\Rightarrow$ $d(x)=\sqrt{x^2+[(x-1)^2]^2}=\sqrt{x^2+(x-1)^4}$. When $d'=0$, d will be minimized and equivalently, $s=d^2$ will be minimized, so we will use Newton's method with $f=s'$ and $f'=s''$.



$f(x)=2x+4(x-1)^3$ $\Rightarrow$ $f'(x)=2+12(x-1)^2$, so $x_{n+1}=x_n-\dfrac{2x_n+4(x_n-1)^3}{2+12(x_n-1)^2}$. Try $x_1=0.5$ $\Rightarrow$ $x_2=0.4$, $x_3\approx 0.410127$, $x_4\approx 0.410245\approx x_5$. Now $d(0.410245)\approx 0.537841$ is the minimum distance and the point on the parabola is $(0.410245, 0.347810)$, correct to six decimal places.

40. Let the radius of the circle be r. Using $s=r\theta$, we have $5=r\theta$ and so $r=5/\theta$. From the Law of Cosines we get $4^2=r^2+r^2-2\cdot r\cdot r\cdot\cos\theta$ $\Leftrightarrow$ $16=2r^2(1-\cos\theta)=2(5/\theta)^2(1-\cos\theta)$. Multiplying by θ^2 gives $16\theta^2=50(1-\cos\theta)$, so we take $f(\theta)=16\theta^2+50\cos\theta-50$ and $f'(\theta)=32\theta-50\sin\theta$. The formula for Newton's method is $\theta_{n+1}=\theta_n-\dfrac{16\theta_n^2+50\cos\theta_n-50}{32\theta_n-50\sin\theta_n}$. From the graph of f, we can use $\theta_1=2.2$, giving us $\theta_2\approx 2.2662$, $\theta_3\approx 2.2622\approx\theta_4$. So correct to four decimal places, the angle is 2.2622 radians $\approx 130^\circ$.



41. In this case, $A=18{,}000$, $R=375$, and $n=5(12)=60$. So the formula $A=\dfrac{R}{i}[1-(1+i)^{-n}]$ becomes $18{,}000=\dfrac{375}{x}[1-(1+x)^{-60}]$ $\Leftrightarrow$ $48x=1-(1+x)^{-60}$ [multiply each term by $(1+x)^{60}$] $\Leftrightarrow$ $48x(1+x)^{60}-(1+x)^{60}+1=0$. Let the LHS be called $f(x)$, so that

$$\begin{aligned} f'(x) &= 48x(60)(1+x)^{59}+48(1+x)^{60}-60(1+x)^{59} \\ &= 12(1+x)^{59}[4x(60)+4(1+x)-5]=12(1+x)^{59}(244x-1) \end{aligned}$$

$x_{n+1}=x_n-\dfrac{48x_n(1+x_n)^{60}-(1+x_n)^{60}+1}{12(1+x_n)^{59}(244x_n-1)}$. An interest rate of 1% per month seems like a reasonable estimate for $x=i$. So let $x_1=1\%=0.01$, and we get $x_2\approx 0.0082202$, $x_3\approx 0.0076802$, $x_4\approx 0.0076291$, $x_5\approx 0.0076286\approx x_6$. Thus, the dealer is charging a monthly interest rate of 0.76286% (or 9.55% per year, compounded monthly).

42. (a) $p(x)=x^5-(2+r)x^4+(1+2r)x^3-(1-r)x^2+2(1-r)x+r-1$ $\Rightarrow$ $p'(x)=5x^4-4(2+r)x^3+3(1+2r)x^2-2(1-r)x+2(1-r)$. So we use

$$x_{n+1}=x_n-\frac{x_n^5-(2+r)x_n^4+(1+2r)x_n^3-(1-r)x_n^2+2(1-r)x_n+r-1}{5x_n^4-4(2+r)x_n^3+3(1+2r)x_n^2-2(1-r)x_n+2(1-r)}.$$

We substitute in the value $r\approx 3.04042\times 10^{-6}$ in order to evaluate the approximations numerically. The libration point L_1 is slightly less than 1 AU from the sun, so we take $x_1=0.95$ as our first approximation, and get $x_2\approx 0.96682$,

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$x_3 \approx 0.97770$, $x_4 \approx 0.98451$, $x_5 \approx 0.98830$, $x_6 \approx 0.98976$, $x_7 \approx 0.98998$, $x_8 \approx 0.98999 \approx x_9$. So, to five decimal places, L_1 is located 0.98999 AU from the sun (or 0.01001 AU from the earth).

(b) In this case we use Newton's method with the function

$p(x) - 2rx^2 = x^5 - (2+r)x^4 + (1+2r)x^3 - (1+r)x^2 + 2(1-r)x + r - 1 \quad \Rightarrow$

$\left[p(x) - 2rx^2\right]' = 5x^4 - 4(2+r)x^3 + 3(1+2r)x^2 - 2(1+r)x + 2(1-r)$. So

$x_{n+1} = x_n - \dfrac{x_n^5 - (2+r)x_n^4 + (1+2r)x_n^3 - (1+r)x_n^2 + 2(1-r)x_n + r - 1}{5x_n^4 - 4(2+r)x_n^3 + 3(1+2r)x_n^2 - 2(1+r)x_n + 2(1-r)}$. Again, we substitute

$r \approx 3.04042 \times 10^{-6}$. L_2 is slightly more than 1 AU from the sun and, judging from the result of part (a), probably less than 0.02 AU from earth. So we take $x_1 = 1.02$ and get $x_2 \approx 1.01422$, $x_3 \approx 1.01118$, $x_4 \approx 1.01018$, $x_5 \approx 1.01008 \approx x_6$. So, to five decimal places, L_2 is located 1.01008 AU from the sun (or 0.01008 AU from the earth).

4.9 Antiderivatives

1. (a) $f(x) = 6 \quad \Rightarrow \quad F(x) = 6x$ is an antiderivative.

(b) $g(t) = 3t^2 \quad \Rightarrow \quad G(t) = 3\,\dfrac{t^{2+1}}{2+1} = t^3$ is an antiderivative.

2. (a) $f(x) = 2x = 2x^1 \quad \Rightarrow \quad F(x) = 2\,\dfrac{x^{1+1}}{1+1} = x^2$ is an antiderivative.

(b) $g(x) = -1/x^2 = -x^{-2} \quad \Rightarrow \quad G(x) = -\dfrac{x^{-2+1}}{-2+1} = x^{-1} = 1/x$ is an antiderivative.

3. (a) $h(q) = \cos q \quad \Rightarrow \quad H(q) = \sin q$ is an antiderivative.

(b) $f(x) = e^x \quad \Rightarrow \quad F(x) = e^x$ is an antiderivative.

4. (a) $g(t) = 1/t \quad \Rightarrow \quad G(t) = \ln|t|$ is an antiderivative.

(b) $r(\theta) = \sec^2\theta \quad \Rightarrow \quad R(\theta) = \tan\theta$ is an antiderivative.

5. $f(x) = 4x + 7 = 4x^1 + 7 \quad \Rightarrow \quad F(x) = 4\dfrac{x^{1+1}}{1+1} + 7x + C = 2x^2 + 7x + C$

Check: $F'(x) = 2(2x) + 7 + 0 = 4x + 7 = f(x)$

6. $f(x) = x^2 - 3x + 2 \quad \Rightarrow \quad F(x) = \dfrac{x^3}{3} - 3\dfrac{x^2}{2} + 2x + C = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C$

Check: $F'(x) = \frac{1}{3}(3x^2) - \frac{3}{2}(2x) + 2 + 0 = x^2 - 3x + 2 = f(x)$

7. $f(x) = 2x^3 - \frac{2}{3}x^2 + 5x \quad \Rightarrow \quad F(x) = 2\,\dfrac{x^{3+1}}{3+1} - \dfrac{2}{3}\dfrac{x^{2+1}}{2+1} + 5\,\dfrac{x^{1+1}}{1+1} = \frac{1}{2}x^4 - \frac{2}{9}x^3 + \frac{5}{2}x^2 + C$

Check: $F'(x) = \frac{1}{2}(4x^3) - \frac{2}{9}(3x^2) + \frac{5}{2}(2x) + 0 = 2x^3 - \frac{2}{3}x^2 + 5x = f(x)$

8. $f(x) = 6x^5 - 8x^4 - 9x^2 \quad \Rightarrow \quad F(x) = 6\,\dfrac{x^6}{6} - 8\,\dfrac{x^5}{5} - 9\,\dfrac{x^3}{3} + C = x^6 - \frac{8}{5}x^5 - 3x^3 + C$

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9. $f(x) = x(12x+8) = 12x^2 + 8x \quad \Rightarrow \quad F(x) = 12\,\frac{x^3}{3} + 8\,\frac{x^2}{2} + C = 4x^3 + 4x^2 + C$

10. $f(x) = (x-5)^2 = x^2 - 10x + 25 \quad \Rightarrow \quad F(x) = \frac{x^3}{3} - 10\frac{x^2}{2} + 25x + C = \frac{1}{3}x^3 - 5x^2 + 25x + C$

11. $g(x) = 4x^{-2/3} - 2x^{5/3} \quad \Rightarrow \quad G(x) = 4(3x^{1/3}) - 2\left(\frac{3}{8}x^{8/3}\right) + C = 12x^{1/3} - \frac{3}{4}x^{8/3} + C$

12. $h(z) = 3z^{0.8} + z^{-2.5} \quad \Rightarrow \quad H(z) = 3\,\frac{z^{1.8}}{1.8} + \frac{z^{-1.5}}{-1.5} = \frac{5}{3}\,z^{1.8} - \frac{2}{3}\,z^{-1.5} + C$

13. $f(x) = 3\sqrt{x} - 2\sqrt[3]{x} = 3x^{1/2} - 2x^{1/3} \quad \Rightarrow \quad F(x) = 3\left(\frac{2}{3}x^{3/2}\right) - 2\left(\frac{3}{4}x^{4/3}\right) + C = 2x^{3/2} - \frac{3}{2}x^{4/3} + C$

14. $g(x) = \sqrt{x}\,(2 - x + 6x^2) = 2x^{1/2} - x^{3/2} + 6x^{5/2} \quad \Rightarrow$

$G(x) = 2\,\frac{x^{3/2}}{3/2} - \frac{x^{5/2}}{5/2} + 6\,\frac{x^{7/2}}{7/2} + C = \frac{4}{3}\,x^{3/2} - \frac{2}{5}\,x^{5/2} + \frac{12}{7}\,x^{7/2} + C$

15. $f(t) = \dfrac{2t - 4 + 3\sqrt{t}}{\sqrt{t}} = 2t^{1/2} - 4t^{-1/2} + 3 \quad \Rightarrow \quad F(t) = 2\,\frac{t^{3/2}}{3/2} - 4\,\frac{t^{1/2}}{1/2} + 3t + C = \frac{4}{3}\,t^{3/2} - 8t^{1/2} + 3t + C$

16. $f(x) = \sqrt[4]{5} + \sqrt[4]{x} = \sqrt[4]{5} + x^{1/4} \quad \Rightarrow \quad F(x) = \sqrt[4]{5}\,x + \frac{x^{5/4}}{5/4} + C = \sqrt[4]{5}\,x + \frac{4}{5}\,x^{5/4} + C$

17. $f(x) = \dfrac{2}{5x} - \dfrac{3}{x^2} = \frac{2}{5}\left(\frac{1}{x}\right) - 3x^{-2}$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} \frac{2}{5}\ln(-x) + \frac{3}{x} + C_2 & \text{if } x < 0 \\ \frac{2}{5}\ln x + \frac{3}{x} + C_1 & \text{if } x > 0 \end{cases}$

See Example 1(b) for a similar problem.

18. $f(x) = \dfrac{5x^2 - 6x + 4}{x^2},\ x > 0 \quad \Rightarrow \quad f(x) = 5 - 6x^{-1} + 4x^{-2},\ x > 0 \quad \Rightarrow$

$F(x) = 5x - 6\ln x + 4(-x^{-1}) + C = 5x - 6\ln x - \dfrac{4}{x} + C,\ x > 0$

19. $g(t) = 7e^t - e^3 \quad \Rightarrow \quad G(t) = 7e^t - e^3 t + C$

20. $f(x) = \dfrac{10}{x^6} - 2e^x + 3$ has domain $(-\infty, 0) \cup (0, \infty)$, so $F(x) = \begin{cases} -\frac{2}{x^5} - 2e^x + 3x + C_1 & \text{if } x < 0 \\ -\frac{2}{x^5} - 2e^x + 3x + C_2 & \text{if } x > 0 \end{cases}$

21. $f(\theta) = 2\sin\theta - 3\sec\theta\tan\theta \quad \Rightarrow \quad F(\theta) = -2\cos\theta - 3\sec\theta + C_n$ on the interval $\left(n\pi - \dfrac{\pi}{2}, n\pi + \dfrac{\pi}{2}\right)$.

22. $h(x) = \sec^2 x + \pi\cos x \quad \Rightarrow \quad H(x) = \tan x + \pi\sin x + C_n$ on the interval $\left(n\pi - \dfrac{\pi}{2}, n\pi + \dfrac{\pi}{2}\right)$.

23. $f(r) = \dfrac{4}{1+r^2} - \sqrt[5]{r^4} = 4\left(\dfrac{1}{1+r^2}\right) - r^{4/5} \quad \Rightarrow \quad F(r) = 4\arctan r - \dfrac{5}{9}r^{9/5} + C$

24. $g(v) = 2\cos v - \dfrac{3}{\sqrt{1-v^2}} \quad \Rightarrow \quad G(v) = 2\sin v - 3\sin^{-1} v + C$

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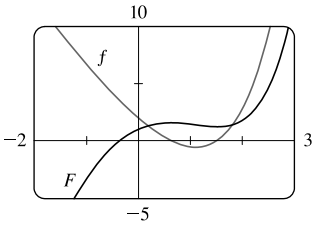
25. $f(x) = 2^x + 4\sinh x \quad \Rightarrow \quad F(x) = \dfrac{2^x}{\ln 2} + 4\cosh x + C$

26. $f(x) = \dfrac{2x^2+5}{x^2+1} = \dfrac{2(x^2+1)+3}{x^2+1} = 2 + \dfrac{3}{x^2+1} \quad \Rightarrow \quad F(x) = 2x + 3\tan^{-1} x + C$

27. $f(x) = 2e^x - 6x \quad \Rightarrow \quad F(x) = 2e^x - 6 \cdot \dfrac{x^2}{2} + C = 2e^x - 3x^2 + C.$

$F(0) = 1 \quad \Rightarrow \quad 2e^0 - 3(0)^2 + C = 1 \quad \Rightarrow \quad C = -1$, so $F(x) = 2e^x - 3x^2 - 1$.

The graph confirms our answer since $f(x) = 0$ when F has a local maximum and when F has a local minimum, f is positive when F is increasing, and f is negative when F is decreasing.

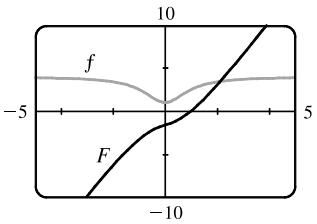


28. $f(x) = 4 - 3\left(1+x^2\right)^{-1} = 4 - \dfrac{3}{1+x^2} \quad \Rightarrow \quad F(x) = 4x - 3\tan^{-1} x + C.$

$F(1) = 0 \quad \Rightarrow \quad 4 - 3\left(\frac{\pi}{4}\right) + C = 0 \quad \Rightarrow \quad C = \frac{3\pi}{4} - 4$, so

$F(x) = 4x - 3\tan^{-1} x + \frac{3\pi}{4} - 4$. Note that f is positive and F is increasing on $\mathbb{R}$.

Also, f has smaller values where the slopes of the tangent lines of F are smaller.



29. $f''(x) = 24x \quad \Rightarrow \quad f'(x) = 24\left(\dfrac{x^2}{2}\right) + C = 12x^2 + C \quad \Rightarrow \quad f(x) = 12\left(\dfrac{x^3}{3}\right) + Cx + D = 4x^3 + Cx + D$

30. $f''(t) = t^2 - 4 \quad \Rightarrow \quad f'(t) = \dfrac{t^3}{3} - 4t + C \quad \Rightarrow \quad f(t) = \dfrac{1}{3}\left(\dfrac{t^4}{4}\right) - 4\left(\dfrac{t^2}{2}\right) + Ct + D = \dfrac{1}{12}t^4 - 2t^2 + Ct + D$

31. $f''(x) = 4x^3 + 24x - 1 \quad \Rightarrow \quad f'(x) = 4\left(\dfrac{x^4}{4}\right) + 24\left(\dfrac{x^2}{2}\right) - x + C = x^4 + 12x^2 - x + C \quad \Rightarrow$

$f(x) = \dfrac{x^5}{5} + 12\left(\dfrac{x^3}{3}\right) - \dfrac{x^2}{2} + Cx + D = \dfrac{1}{5}x^5 + 4x^3 - \dfrac{1}{2}x^2 + Cx + D$

32. $f''(x) = 6x - x^4 + 3x^5 \quad \Rightarrow \quad f'(x) = 6\left(\dfrac{x^2}{2}\right) - \dfrac{x^5}{5} + 3\left(\dfrac{x^6}{6}\right) + C = 3x^2 - \dfrac{1}{5}x^5 + \dfrac{1}{2}x^6 + C \quad \Rightarrow$

$f(x) = 3\left(\dfrac{x^3}{3}\right) - \dfrac{1}{5}\left(\dfrac{x^6}{6}\right) + \dfrac{1}{2}\left(\dfrac{x^7}{7}\right) + Cx + D = x^3 - \dfrac{1}{30}x^6 + \dfrac{1}{14}x^7 + Cx + D$

33. $f''(x) = 2x + 3e^x \quad \Rightarrow \quad f'(x) = x^2 + 3e^x + C \quad \Rightarrow \quad f(x) = \frac{1}{3}x^3 + 3e^x + Cx + D$

34. $f''(x) = 1/x^2 = x^{-2} \quad \Rightarrow \quad f'(x) = \begin{cases} -1/x + C_1 & \text{if } x < 0 \\ -1/x + C_2 & \text{if } x > 0 \end{cases} \quad \Rightarrow \quad f(x) = \begin{cases} -\ln(-x) + C_1 x + D_1 & \text{if } x < 0 \\ -\ln x + C_2 x + D_2 & \text{if } x > 0 \end{cases}$

35. $f'''(t) = 12 + \sin t \quad \Rightarrow \quad f''(t) = 12t - \cos t + C_1 \quad \Rightarrow \quad f'(t) = 6t^2 - \sin t + C_1 t + D \quad \Rightarrow$

$f(t) = 2t^3 + \cos t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

36. $f'''(t) = \sqrt{t} - 2\cos t = t^{1/2} - 2\cos t \quad \Rightarrow \quad f''(t) = \frac{2}{3}t^{3/2} - 2\sin t + C_1 \quad \Rightarrow \quad f'(t) = \frac{4}{15}t^{5/2} + 2\cos t + C_1 t + D \quad \Rightarrow$

$f(t) = \frac{8}{105}t^{7/2} + 2\sin t + Ct^2 + Dt + E$, where $C = \frac{1}{2}C_1$.

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37. $f'(x) = 8x^3 + \dfrac{1}{x},\ x > 0 \quad \Rightarrow \quad f(x) = 2x^4 + \ln x + C. \quad f(1) = 2(1)^4 + \ln 1 + C = 2 + C$ and $f(1) = -3 \quad \Rightarrow$ $C = -5$, so $f(x) = 2x^4 + \ln x - 5$.

38. $f'(x) = \sqrt{x} - 2 \quad \Rightarrow \quad f(x) = \frac{2}{3}x^{3/2} - 2x + C$.

$f(9) = \frac{2}{3}(9)^{3/2} - 2(9) + C = 18 - 18 + C = C$ and $f(9) = 4 \quad \Rightarrow \quad C = 4$, so $f(x) = \frac{2}{3}x^{3/2} - 2x + 4$.

39. $f'(t) = \dfrac{4}{1+t^2} \quad \Rightarrow \quad f(t) = 4\arctan t + C. \quad f(1) = 4\left(\dfrac{\pi}{4}\right) + C$ and $f(1) = 0 \quad \Rightarrow \quad \pi + C = 0 \quad \Rightarrow \quad C = -\pi$, so $f(t) = 4\arctan t - \pi$.

40. $f'(t) = t + \dfrac{1}{t^3},\ t > 0 \quad \Rightarrow \quad f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + C. \quad f(1) = \dfrac{1}{2} - \dfrac{1}{2} + C$ and $f(1) = 6 \quad \Rightarrow \quad C = 6$, so $f(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2t^2} + 6$.

41. $f'(x) = 5x^{2/3} \quad \Rightarrow \quad f(x) = 5\left(\frac{3}{5}x^{5/3}\right) + C = 3x^{5/3} + C$.

$f(8) = 3 \cdot 32 + C$ and $f(8) = 21 \quad \Rightarrow \quad 96 + C = 21 \quad \Rightarrow \quad C = -75$, so $f(x) = 3x^{5/3} - 75$.

42. $f'(x) = \dfrac{x+1}{\sqrt{x}} = x^{1/2} + x^{-1/2} \quad \Rightarrow \quad f(x) = \frac{2}{3}x^{3/2} + 2x^{1/2} + C$.

$f(1) = \frac{2}{3} + 2 + C = \frac{8}{3} + C$ and $f(1) = 5 \quad \Rightarrow \quad C = 5 - \frac{8}{3} = \frac{7}{3}$, so $f(x) = \frac{2}{3}x^{3/2} + 2\sqrt{x} + \frac{7}{3}$.

43. $f'(t) = \sec t\,(\sec t + \tan t) = \sec^2 t + \sec t\,\tan t, -\frac{\pi}{2} < t < \frac{\pi}{2} \quad \Rightarrow \quad f(t) = \tan t + \sec t + C. \quad f\left(\frac{\pi}{4}\right) = 1 + \sqrt{2} + C$ and $f\left(\frac{\pi}{4}\right) = -1 \quad \Rightarrow \quad 1 + \sqrt{2} + C = -1 \quad \Rightarrow \quad C = -2 - \sqrt{2}$, so $f(t) = \tan t + \sec t - 2 - \sqrt{2}$.

Note: The fact that f is defined and continuous on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ means that we have only one constant of integration.

44. $f'(t) = 3^t - \dfrac{3}{t} \quad \Rightarrow \quad f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + C & \text{if} \quad t < 0 \\ 3^t/\ln 3 - 3\ln t + D & \text{if} \quad t > 0 \end{cases}$

$f(-1) = \dfrac{1}{3\ln 3} - 3\ln 1 + C$ and $f(-1) = 1 \quad \Rightarrow \quad C = 1 - \dfrac{1}{3\ln 3}$.

$f(1) = \dfrac{3}{\ln 3} - 3\ln 1 + D$ and $f(1) = 2 \quad \Rightarrow \quad D = 2 - \dfrac{3}{\ln 3}$.

Thus, $f(t) = \begin{cases} 3^t/\ln 3 - 3\ln(-t) + 1 - 1/(3\ln 3) & \text{if} \quad t < 0 \\ 3^t/\ln 3 - 3\ln t + 2 - 3/\ln 3 & \text{if} \quad t > 0 \end{cases}$

45. $f''(x) = -2 + 12x - 12x^2 \quad \Rightarrow \quad f'(x) = -2x + 6x^2 - 4x^3 + C$. $f'(0) = C$ and $f'(0) = 12 \quad \Rightarrow \quad C = 12$, so $f'(x) = -2x + 6x^2 - 4x^3 + 12$ and hence, $f(x) = -x^2 + 2x^3 - x^4 + 12x + D$. $f(0) = D$ and $f(0) = 4 \quad \Rightarrow \quad D = 4$, so $f(x) = -x^2 + 2x^3 - x^4 + 12x + 4$.

46. $f''(x) = 8x^3 + 5 \quad \Rightarrow \quad f'(x) = 2x^4 + 5x + C$. $f'(1) = 2 + 5 + C$ and $f'(1) = 8 \quad \Rightarrow \quad C = 1$, so $f'(x) = 2x^4 + 5x + 1$. $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x + D$. $f(1) = \frac{2}{5} + \frac{5}{2} + 1 + D = D + \frac{39}{10}$ and $f(1) = 0 \quad \Rightarrow \quad D = -\frac{39}{10}$, so $f(x) = \frac{2}{5}x^5 + \frac{5}{2}x^2 + x - \frac{39}{10}$.

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47. $f''(\theta) = \sin\theta + \cos\theta \;\Rightarrow\; f'(\theta) = -\cos\theta + \sin\theta + C$. $f'(0) = -1 + C$ and $f'(0) = 4 \;\Rightarrow\; C = 5$, so $f'(\theta) = -\cos\theta + \sin\theta + 5$ and hence, $f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$. $f(0) = -1 + D$ and $f(0) = 3 \;\Rightarrow\; D = 4$, so $f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$.

48. $f''(t) = t^2 + \dfrac{1}{t^2} = t^2 + t^{-2},\ t > 0 \;\Rightarrow\; f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + C$. $f'(1) = \frac{1}{3} - 1 + C$ and $f'(1) = 2 \;\Rightarrow\; C - \frac{2}{3} = 2 \;\Rightarrow\; C = \frac{8}{3}$, so $f'(t) = \frac{1}{3}t^3 - \dfrac{1}{t} + \frac{8}{3}$ and hence, $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + D$. $f(2) = \frac{4}{3} - \ln 2 + \frac{16}{3} + D$ and $f(2) = 3 \;\Rightarrow\; \frac{20}{3} - \ln 2 + D = 3 \;\Rightarrow\; D = \ln 2 - \frac{11}{3}$, so $f(t) = \frac{1}{12}t^4 - \ln t + \frac{8}{3}t + \ln 2 - \frac{11}{3}$.

49. $f''(x) = 4 + 6x + 24x^2 \;\Rightarrow\; f'(x) = 4x + 3x^2 + 8x^3 + C \;\Rightarrow\; f(x) = 2x^2 + x^3 + 2x^4 + Cx + D$. $f(0) = D$ and $f(0) = 3 \;\Rightarrow\; D = 3$, so $f(x) = 2x^2 + x^3 + 2x^4 + Cx + 3$. $f(1) = 8 + C$ and $f(1) = 10 \;\Rightarrow\; C = 2$, so $f(x) = 2x^2 + x^3 + 2x^4 + 2x + 3$.

50. $f''(x) = x^3 + \sinh x \;\Rightarrow\; f'(x) = \frac{1}{4}x^4 + \cosh x + C \;\Rightarrow\; f(x) = \frac{1}{20}x^5 + \sinh x + Cx + D$. $f(0) = D$ and $f(0) = 1 \;\Rightarrow\; D = 1$, so $f(x) = \frac{1}{20}x^5 + \sinh x + Cx + 1$. $f(2) = \frac{32}{20} + \sinh 2 + 2C + 1$ and $f(2) = 2.6 \;\Rightarrow\; \sinh 2 + 2C = 0 \;\Rightarrow\; C = -\frac{1}{2}\sinh 2$, so $f(x) = \frac{1}{20}x^5 + \sinh x - \frac{1}{2}(\sinh 2)x + 1$.

51. $f''(x) = e^x - 2\sin x \;\Rightarrow\; f'(x) = e^x + 2\cos x + C \;\Rightarrow\; f(x) = e^x + 2\sin x + Cx + D$. $f(0) = 1 + 0 + D$ and $f(0) = 3 \;\Rightarrow\; D = 2$, so $f(x) = e^x + 2\sin x + Cx + 2$. $f\left(\frac{\pi}{2}\right) = e^{\pi/2} + 2 + \frac{\pi}{2}C + 2$ and $f\left(\frac{\pi}{2}\right) = 0 \;\Rightarrow\; e^{\pi/2} + 4 + \frac{\pi}{2}C = 0 \;\Rightarrow\; \frac{\pi}{2}C = -e^{\pi/2} - 4 \;\Rightarrow\; C = -\frac{2}{\pi}(e^{\pi/2} + 4)$, so $f(x) = e^x + 2\sin x - \frac{2}{\pi}(e^{\pi/2} + 4)x + 2$.

52. $f''(t) = \sqrt[3]{t} - \cos t = t^{1/3} - \cos t \;\Rightarrow\; f'(t) = \frac{3}{4}t^{4/3} - \sin t + C \;\Rightarrow\; f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + D$. $f(0) = 0 + 1 + 0 + D$ and $f(0) = 2 \;\Rightarrow\; D = 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + Ct + 1$. $f(1) = \frac{9}{28} + \cos 1 + C + 1$ and $f(1) = 2 \;\Rightarrow\; C = 2 - \frac{9}{28} - \cos 1 - 1 = \frac{19}{28} - \cos 1$, so $f(t) = \frac{9}{28}t^{7/3} + \cos t + \left(\frac{19}{28} - \cos 1\right)t + 1$.

53. $f''(x) = x^{-2}, x > 0 \;\Rightarrow\; f'(x) = -1/x + C \;\Rightarrow\; f(x) = -\ln|x| + Cx + D = -\ln x + Cx + D$ [since $x > 0$]. $f(1) = 0 \;\Rightarrow\; C + D = 0$ and $f(2) = 0 \;\Rightarrow\; -\ln 2 + 2C + D = 0 \;\Rightarrow\; -\ln 2 + 2C - C = 0$ [since $D = -C$] $\Rightarrow\; -\ln 2 + C = 0 \;\Rightarrow\; C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

54. $f'''(x) = \cos x \;\Rightarrow\; f''(x) = \sin x + C$. $f''(0) = C$ and $f''(0) = 3 \;\Rightarrow\; C = 3$. $f''(x) = \sin x + 3 \;\Rightarrow\; f'(x) = -\cos x + 3x + D$. $f'(0) = -1 + D$ and $f'(0) = 2 \;\Rightarrow\; D = 3$. $f'(x) = -\cos x + 3x + 3 \;\Rightarrow\; f(x) = -\sin x + \frac{3}{2}x^2 + 3x + E$. $f(0) = E$ and $f(0) = 1 \;\Rightarrow\; E = 1$. Thus, $f(x) = -\sin x + \frac{3}{2}x^2 + 3x + 1$.

55. "The slope of its tangent line at $(x, f(x))$ is $3 - 4x$" means that $f'(x) = 3 - 4x$, so $f(x) = 3x - 2x^2 + C$. "The graph of f passes through the point $(2, 5)$" means that $f(2) = 5$, but $f(2) = 3(2) - 2(2)^2 + C$, so $5 = 6 - 8 + C \;\Rightarrow\; C = 7$. Thus, $f(x) = 3x - 2x^2 + 7$ and $f(1) = 3 - 2 + 7 = 8$.

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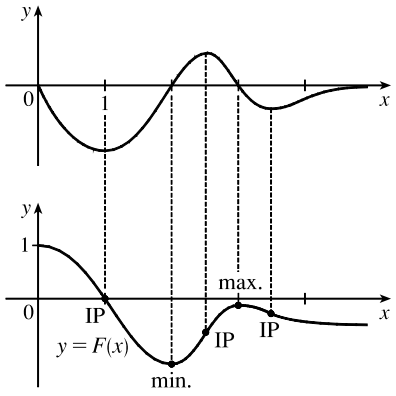
56. $f'(x) = x^3 \;\Rightarrow\; f(x) = \frac{1}{4}x^4 + C$. $x + y = 0 \;\Rightarrow\; y = -x \;\Rightarrow\; m = -1$. Now $m = f'(x) \;\Rightarrow\; -1 = x^3 \;\Rightarrow\; x = -1 \;\Rightarrow\; y = 1$ (from the equation of the tangent line), so $(-1, 1)$ is a point on the graph of f. From f, $1 = \frac{1}{4}(-1)^4 + C \;\Rightarrow\; C = \frac{3}{4}$. Therefore, the function is $f(x) = \frac{1}{4}x^4 + \frac{3}{4}$.

57. b is the antiderivative of f. For small x, f is negative, so the graph of its antiderivative must be decreasing. But both a and c are increasing for small x, so only b can be f's antiderivative. Also, f is positive where b is increasing, which supports our conclusion.

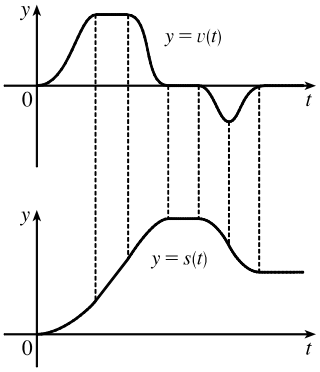
58. We know right away that c cannot be f's antiderivative, since the slope of c is not zero at the x-value where $f = 0$. Now f is positive when a is increasing and negative when a is decreasing, so a is the antiderivative of f.

59.



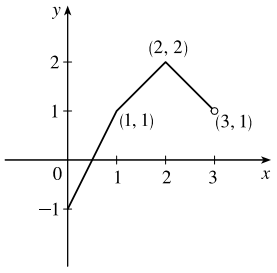
The graph of F must start at $(0, 1)$. Where the given graph, $y = f(x)$, has a local minimum or maximum, the graph of F will have an inflection point.
Where f is negative (positive), F is decreasing (increasing).
Where f changes from negative to positive, F will have a minimum.
Where f changes from positive to negative, F will have a maximum.
Where f is decreasing (increasing), F is concave downward (upward).

60.



Where v is positive (negative), s is increasing (decreasing).
Where v is increasing (decreasing), s is concave upward (downward).
Where v is horizontal (a steady velocity), s is linear.

61.



$$f'(x) = \begin{cases} 2 & \text{if } 0 \le x < 1 \\ 1 & \text{if } 1 < x < 2 \\ -1 & \text{if } 2 < x < 3 \end{cases} \;\Rightarrow\; f(x) = \begin{cases} 2x + C & \text{if } 0 \le x < 1 \\ x + D & \text{if } 1 < x < 2 \\ -x + E & \text{if } 2 < x < 3 \end{cases}$$

$f(0) = -1 \;\Rightarrow\; 2(0) + C = -1 \;\Rightarrow\; C = -1$. Starting at the point $(0, -1)$ and moving to the right on a line with slope 2 gets us to the point $(1, 1)$.

The slope for $1 < x < 2$ is 1, so we get to the point $(2, 2)$. Here we have used the fact that f is continuous. We can include the

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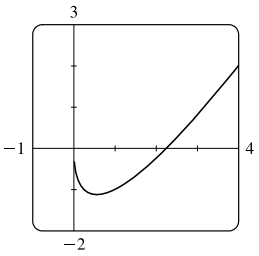
point $x = 1$ on either the first or the second part of f. The line connecting $(1, 1)$ to $(2, 2)$ is $y = x$, so $D = 0$. The slope for $2 < x < 3$ is -1, so we get to $(3, 1)$. $f(2) = 2 \Rightarrow -2 + E = 2 \Rightarrow E = 4$. Thus,

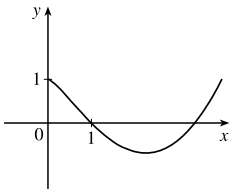
$$f(x) = \begin{cases} 2x - 1 & \text{if } 0 \le x \le 1 \\ x & \text{if } 1 < x < 2 \\ -x + 4 & \text{if } 2 \le x < 3 \end{cases}$$

Note that $f'(x)$ does not exist at $x = 1$, 2, or 3.

62. (a)

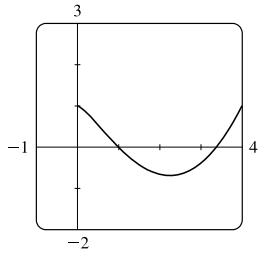


(b) Since $F(0) = 1$, we can start our graph at $(0, 1)$. f has a minimum at about $x = 0.5$, so its derivative is zero there. f is decreasing on $(0, 0.5)$, so its derivative is negative and hence, F is CD on $(0, 0.5)$ and has an IP at $x \approx 0.5$. On $(0.5, 2.2)$, f is negative and increasing (f' is positive), so F is decreasing and CU. On $(2.2, \infty)$, f is positive and increasing, so F is increasing and CU.



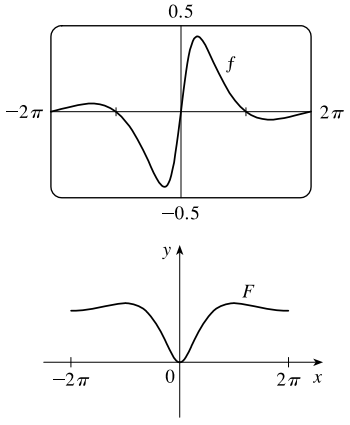
(c) $f(x) = 2x - 3\sqrt{x} \Rightarrow F(x) = x^2 - 3 \cdot \frac{2}{3}x^{3/2} + C$.
$F(0) = C$ and $F(0) = 1 \Rightarrow C = 1$, so $F(x) = x^2 - 2x^{3/2} + 1$.

(d)



63. $f(x) = \dfrac{\sin x}{1 + x^2}$, $-2\pi \le x \le 2\pi$

Note that the graph of f is one of an odd function, so the graph of F will be one of an even function.



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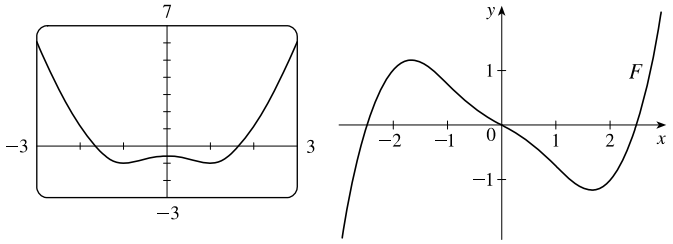
64. $f(x) = \sqrt{x^4 - 2x^2 + 2} - 2,\ -3 \le x \le 3$

Note that the graph of f is one of an even function, so the graph of F will be one of an odd function.



65. $v(t) = s'(t) = 2\cos t + 4\sin t \ \Rightarrow\ s(t) = 2\sin t - 4\cos t + C$. $s(0) = -4 + C$ and $s(0) = 3 \ \Rightarrow\ C = 7$, so $s(t) = 2\sin t - 4\cos t + 7$.

66. $v(t) = s'(t) = t^2 - 3\sqrt{t} = t^2 - 3t^{1/2} \ \Rightarrow\ s(t) = \frac{1}{3}t^3 - 2t^{3/2} + C$. $s(4) = \frac{64}{3} - 16 + C$ and $s(4) = 8 \ \Rightarrow$ $C = 8 - \frac{64}{3} + 16 = \frac{8}{3}$, so $s(t) = \frac{1}{3}t^3 - 2t^{3/2} + \frac{8}{3}$.

67. $a(t) = v'(t) = 2t + 1 \ \Rightarrow\ v(t) = t^2 + t + C$. $v(0) = C$ and $v(0) = -2 \ \Rightarrow\ C = -2$, so $v(t) = t^2 + t - 2$ and $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + D$. $s(0) = D$ and $s(0) = 3 \ \Rightarrow\ D = 3$, so $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$.

68. $a(t) = v'(t) = 3\cos t - 2\sin t \ \Rightarrow\ v(t) = 3\sin t + 2\cos t + C$. $v(0) = 2 + C$ and $v(0) = 4 \ \Rightarrow\ C = 2$, so $v(t) = 3\sin t + 2\cos t + 2$ and $s(t) = -3\cos t + 2\sin t + 2t + D$. $s(0) = -3 + D$ and $s(0) = 0 \ \Rightarrow\ D = 3$, so $s(t) = -3\cos t + 2\sin t + 2t + 3$.

69. $a(t) = v'(t) = \sin t - \cos t \ \Rightarrow\ v(t) = s'(t) = -\cos t - \sin t + C \ \Rightarrow\ s(t) = -\sin t + \cos t + Ct + D$.

$s(0) = 1 + D = 0$ and $s(\pi) = -1 + C\pi + D = 6 \ \Rightarrow\ D = -1$ and $C = \dfrac{8}{\pi}$. Thus, $s(t) = -\sin t + \cos t + \dfrac{8}{\pi}t - 1$.

70. $a(t) = t^2 - 4t + 6 \ \Rightarrow\ v(t) = \frac{1}{3}t^3 - 2t^2 + 6t + C \ \Rightarrow\ s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + Ct + D$. $s(0) = D$ and $s(0) = 0 \ \Rightarrow\ D = 0$. $s(1) = \frac{29}{12} + C$ and $s(1) = 20 \ \Rightarrow\ C = \frac{211}{12}$. Thus, $s(t) = \frac{1}{12}t^4 - \frac{2}{3}t^3 + 3t^2 + \frac{211}{12}t$.

71. (a) We first observe that since the stone is dropped 450 m above the ground, $v(0) = 0$ and $s(0) = 450$.

$v'(t) = a(t) = -9.8 \ \Rightarrow\ v(t) = -9.8t + C$. Now $v(0) = 0 \ \Rightarrow\ C = 0$, so $v(t) = -9.8t \ \Rightarrow$ $s(t) = -4.9t^2 + D$. Last, $s(0) = 450 \ \Rightarrow\ D = 450 \ \Rightarrow\ s(t) = 450 - 4.9t^2$.

(b) The stone reaches the ground when $s(t) = 0$. $450 - 4.9t^2 = 0 \ \Rightarrow\ t^2 = 450/4.9 \ \Rightarrow\ t_1 = \sqrt{450/4.9} \approx 9.58$ s.

(c) The velocity with which the stone strikes the ground is $v(t_1) = -9.8\sqrt{450/4.9} \approx -93.9$ m/s.

(d) This is just reworking parts (a) and (b) with $v(0) = -5$. Using $v(t) = -9.8t + C$, $v(0) = -5 \ \Rightarrow\ 0 + C = -5 \ \Rightarrow$ $v(t) = -9.8t - 5$. So $s(t) = -4.9t^2 - 5t + D$ and $s(0) = 450 \ \Rightarrow\ D = 450 \ \Rightarrow\ s(t) = -4.9t^2 - 5t + 450$.

Solving $s(t) = 0$ by using the quadratic formula gives us $t = \left(5 \pm \sqrt{8845}\right)/(-9.8) \ \Rightarrow\ t_1 \approx 9.09$ s.

72. $v'(t) = a(t) = a \ \Rightarrow\ v(t) = at + C$ and $v_0 = v(0) = C \ \Rightarrow\ v(t) = at + v_0 \ \Rightarrow$ $s(t) = \frac{1}{2}at^2 + v_0t + D \ \Rightarrow\ s_0 = s(0) = D \ \Rightarrow\ s(t) = \frac{1}{2}at^2 + v_0t + s_0$

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73. By Exercise 72 with $a = -9.8$, $s(t) = -4.9t^2 + v_0 t + s_0$ and $v(t) = s'(t) = -9.8t + v_0$. So

$[v(t)]^2 = (-9.8t + v_0)^2 = (9.8)^2 t^2 - 19.6 v_0 t + v_0^2 = v_0^2 + 96.04t^2 - 19.6 v_0 t = v_0^2 - 19.6\big(-4.9t^2 + v_0 t\big)$.

But $-4.9t^2 + v_0 t$ is just $s(t)$ without the s_0 term; that is, $s(t) - s_0$. Thus, $[v(t)]^2 = v_0^2 - 19.6\,[s(t) - s_0]$.

74. For the first ball, $s_1(t) = -16t^2 + 48t + 432$ from Example 7. For the second ball, $a(t) = -32 \;\Rightarrow\; v(t) = -32t + C$, but $v(1) = -32(1) + C = 24 \;\Rightarrow\; C = 56$, so $v(t) = -32t + 56 \;\Rightarrow\; s(t) = -16t^2 + 56t + D$, but $s(1) = -16(1)^2 + 56(1) + D = 432 \;\Rightarrow\; D = 392$, and $s_2(t) = -16t^2 + 56t + 392$. The balls pass each other when $s_1(t) = s_2(t) \;\Rightarrow\; -16t^2 + 48t + 432 = -16t^2 + 56t + 392 \;\Leftrightarrow\; 8t = 40 \;\Leftrightarrow\; t = 5$ s.

Another solution: From Exercise 72, we have $s_1(t) = -16t^2 + 48t + 432$ and $s_2(t) = -16t^2 + 24t + 432$. We now want to solve $s_1(t) = s_2(t-1) \;\Rightarrow\; -16t^2 + 48t + 432 = -16(t-1)^2 + 24(t-1) + 432 \;\Rightarrow$ $48t = 32t - 16 + 24t - 24 \;\Rightarrow\; 40 = 8t \;\Rightarrow\; t = 5$ s.

75. Using Exercise 72 with $a = -32$, $v_0 = 0$, and $s_0 = h$ (the height of the cliff), we know that the height at time t is $s(t) = -16t^2 + h$. $v(t) = s'(t) = -32t$ and $v(t) = -120 \;\Rightarrow\; -32t = -120 \;\Rightarrow\; t = 3.75$, so $0 = s(3.75) = -16(3.75)^2 + h \;\Rightarrow\; h = 16(3.75)^2 = 225$ ft.

76. (a) $EIy'' = mg(L-x) + \frac{1}{2}\rho g(L-x)^2 \;\Rightarrow\; EIy' = -\frac{1}{2}mg(L-x)^2 - \frac{1}{6}\rho g(L-x)^3 + C \;\Rightarrow$ $EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + Cx + D$. Since the left end of the board is fixed, we must have $y = y' = 0$ when $x = 0$. Thus, $0 = -\frac{1}{2}mgL^2 - \frac{1}{6}\rho gL^3 + C$ and $0 = \frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4 + D$. It follows that $EIy = \frac{1}{6}mg(L-x)^3 + \frac{1}{24}\rho g(L-x)^4 + \big(\frac{1}{2}mgL^2 + \frac{1}{6}\rho gL^3\big)x - \big(\frac{1}{6}mgL^3 + \frac{1}{24}\rho gL^4\big)$ and

$$f(x) = y = \frac{1}{EI}\big[\tfrac{1}{6}mg(L-x)^3 + \tfrac{1}{24}\rho g(L-x)^4 + \big(\tfrac{1}{2}mgL^2 + \tfrac{1}{6}\rho gL^3\big)x - \big(\tfrac{1}{6}mgL^3 + \tfrac{1}{24}\rho gL^4\big)\big]$$

(b) $f(L) < 0$, so the end of the board is a *distance* approximately $-f(L)$ below the horizontal. From our result in (a), we calculate

$$-f(L) = \frac{-1}{EI}\left[\tfrac{1}{2}mgL^3 + \tfrac{1}{6}\rho gL^4 - \tfrac{1}{6}mgL^3 - \tfrac{1}{24}\rho gL^4\right] = \frac{-1}{EI}\left(\tfrac{1}{3}mgL^3 + \tfrac{1}{8}\rho gL^4\right) = -\frac{gL^3}{EI}\left(\frac{m}{3} + \frac{\rho L}{8}\right)$$

Note: This is positive because g is negative.

77. Marginal cost $= 1.92 - 0.002x = C'(x) \;\Rightarrow\; C(x) = 1.92x - 0.001x^2 + K$. But $C(1) = 1.92 - 0.001 + K = 562 \;\Rightarrow$ $K = 560.081$. Therefore, $C(x) = 1.92x - 0.001x^2 + 560.081 \;\Rightarrow\; C(100) = 742.081$, so the cost of producing 100 items is \$742.08.

78. Let the mass, measured from one end, be $m(x)$. Then $m(0) = 0$ and $\rho = \dfrac{dm}{dx} = x^{-1/2} \;\Rightarrow\; m(x) = 2x^{1/2} + C$ and $m(0) = C = 0$, so $m(x) = 2\sqrt{x}$. Thus, the mass of the 100-centimeter rod is $m(100) = 2\sqrt{100} = 20$ g.

79. Taking the upward direction to be positive we have that for $0 \le t \le 10$ (using the subscript 1 to refer to $0 \le t \le 10$), $a_1(t) = -(9 - 0.9t) = v_1'(t) \;\Rightarrow\; v_1(t) = -9t + 0.45t^2 + v_0$, but $v_1(0) = v_0 = -10 \;\Rightarrow$

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$v_1(t) = -9t + 0.45t^2 - 10 = s_1'(t) \quad \Rightarrow \quad s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + s_0$. But $s_1(0) = 500 = s_0 \quad \Rightarrow$

$s_1(t) = -\frac{9}{2}t^2 + 0.15t^3 - 10t + 500$. $s_1(10) = -450 + 150 - 100 + 500 = 100$, so it takes

more than 10 seconds for the raindrop to fall. Now for $t > 10$, $a(t) = 0 = v'(t) \quad \Rightarrow$

$v(t) = \text{constant} = v_1(10) = -9(10) + 0.45(10)^2 - 10 = -55 \quad \Rightarrow \quad v(t) = -55$.

At 55 m/s, it will take $100/55 \approx 1.8$ s to fall the last 100 m. Hence, the total time is $10 + \frac{100}{55} = \frac{130}{11} \approx 11.8$ s.

80. $v'(t) = a(t) = -22$. The initial velocity is 50 mi/h $= \frac{50 \cdot 5280}{3600} = \frac{220}{3}$ ft/s, so $v(t) = -22t + \frac{220}{3}$.

The car stops when $v(t) = 0 \quad \Leftrightarrow \quad t = \frac{220}{3 \cdot 22} = \frac{10}{3}$. Since $s(t) = -11t^2 + \frac{220}{3}t$, the distance covered is

$s\left(\frac{10}{3}\right) = -11\left(\frac{10}{3}\right)^2 + \frac{220}{3} \cdot \frac{10}{3} = \frac{1100}{9} = 122.\overline{2}$ ft.

81. $a(t) = k$, the initial velocity is 30 mi/h $= 30 \cdot \frac{5280}{3600} = 44$ ft/s, and the final velocity (after 5 seconds) is

50 mi/h $= 50 \cdot \frac{5280}{3600} = \frac{220}{3}$ ft/s. So $v(t) = kt + C$ and $v(0) = 44 \quad \Rightarrow \quad C = 44$. Thus, $v(t) = kt + 44 \quad \Rightarrow$

$v(5) = 5k + 44$. But $v(5) = \frac{220}{3}$, so $5k + 44 = \frac{220}{3} \quad \Rightarrow \quad 5k = \frac{88}{3} \quad \Rightarrow \quad k = \frac{88}{15} \approx 5.87$ ft/s^2.

82. $a(t) = -16 \quad \Rightarrow \quad v(t) = -16t + v_0$ where v_0 is the car's speed (in ft/s) when the brakes were applied. The car stops when

$-16t + v_0 = 0 \quad \Leftrightarrow \quad t = \frac{1}{16}v_0$. Now $s(t) = \frac{1}{2}(-16)t^2 + v_0 t = -8t^2 + v_0 t$. The car travels 200 ft in the time that it takes

to stop, so $s\left(\frac{1}{16}v_0\right) = 200 \quad \Rightarrow \quad 200 = -8\left(\frac{1}{16}v_0\right)^2 + v_0\left(\frac{1}{16}v_0\right) = \frac{1}{32}v_0^2 \quad \Rightarrow \quad v_0^2 = 32 \cdot 200 = 6400 \quad \Rightarrow$

$v_0 = 80$ ft/s $[54.\overline{54}$ mi/h$]$.

83. Let the acceleration be $a(t) = k$ km/h^2. We have $v(0) = 100$ km/h and we can take the initial position $s(0)$ to be 0.

We want the time t_f for which $v(t) = 0$ to satisfy $s(t) < 0.08$ km. In general, $v'(t) = a(t) = k$, so $v(t) = kt + C$,

where $C = v(0) = 100$. Now $s'(t) = v(t) = kt + 100$, so $s(t) = \frac{1}{2}kt^2 + 100t + D$, where $D = s(0) = 0$.

Thus, $s(t) = \frac{1}{2}kt^2 + 100t$. Since $v(t_f) = 0$, we have $kt_f + 100 = 0$ or $t_f = -100/k$, so

$s(t_f) = \frac{1}{2}k\left(-\frac{100}{k}\right)^2 + 100\left(-\frac{100}{k}\right) = 10{,}000\left(\frac{1}{2k} - \frac{1}{k}\right) = -\frac{5{,}000}{k}$. The condition $s(t_f)$ must satisfy is

$-\frac{5{,}000}{k} < 0.08 \quad \Rightarrow \quad -\frac{5{,}000}{0.08} > k \quad [k \text{ is negative}] \quad \Rightarrow \quad k < -62{,}500$ km/h^2, or equivalently,

$k < -\frac{3125}{648} \approx -4.82$ m/s^2.

84. (a) For $0 \le t \le 3$ we have $a(t) = 60t \quad \Rightarrow \quad v(t) = 30t^2 + C \quad \Rightarrow \quad v(0) = 0 = C \quad \Rightarrow \quad v(t) = 30t^2$, so

$s(t) = 10t^3 + C \quad \Rightarrow \quad s(0) = 0 = C \quad \Rightarrow \quad s(t) = 10t^3$. Note that $v(3) = 270$ and $s(3) = 270$.

For $3 < t \le 17$: $a(t) = -g = -32$ ft/s $\quad \Rightarrow \quad v(t) = -32(t-3) + C \quad \Rightarrow \quad v(3) = 270 = C \quad \Rightarrow$

$v(t) = -32(t-3) + 270 \quad \Rightarrow \quad s(t) = -16(t-3)^2 + 270(t-3) + C \quad \Rightarrow \quad s(3) = 270 = C \quad \Rightarrow$

$s(t) = -16(t-3)^2 + 270(t-3) + 270$. Note that $v(17) = -178$ and $s(17) = 914$.

For $17 < t \le 22$: The velocity increases linearly from -178 ft/s to -18 ft/s during this period, so

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$\dfrac{\Delta v}{\Delta t} = \dfrac{-18-(-178)}{22-17} = \dfrac{160}{5} = 32$. Thus, $v(t) = 32(t-17) - 178 \quad \Rightarrow$

$s(t) = 16(t-17)^2 - 178(t-17) + 914$ and $s(22) = 424$ ft.

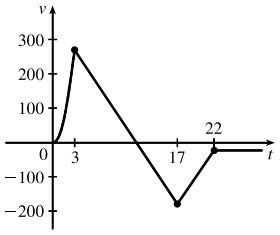
For $t > 22$: $v(t) = -18 \quad \Rightarrow \quad s(t) = -18(t-22) + C$. But $s(22) = 424 = C \quad \Rightarrow \quad s(t) = -18(t-22) + 424$.

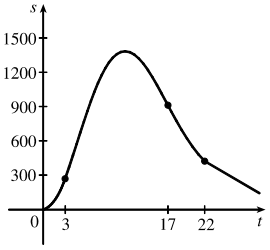
Therefore, until the rocket lands, we have

$$v(t) = \begin{cases} 30t^2 & \text{if } 0 \le t \le 3 \\ -32\,(t-3) + 270 & \text{if } 3 < t \le 17 \\ 32(t-17) - 178 & \text{if } 17 < t \le 22 \\ -18 & \text{if } t > 22 \end{cases}$$

and

$$s(t) = \begin{cases} 10t^3 & \text{if } 0 \le t \le 3 \\ -16(t-3)^2 + 270(t-3) + 270 & \text{if } 3 < t \le 17 \\ 16(t-17)^2 - 178\,(t-17) + 914 & \text{if } 17 < t \le 22 \\ -18(t-22) + 424 & \text{if } t > 22 \end{cases}$$





(b) To find the maximum height, set $v(t)$ on $3 < t \le 17$ equal to 0. $-32(t-3) + 270 = 0 \quad \Rightarrow \quad t_1 = 11.4375$ s and the maximum height is $s(t_1) = -16(t_1 - 3)^2 + 270(t_1 - 3) + 270 = 1409.0625$ ft.

(c) To find the time to land, set $s(t) = -18(t-22) + 424 = 0$. Then $t - 22 = \frac{424}{18} = 23.\overline{5}$, so $t \approx 45.6$ s.

85. (a) First note that $180 \text{ mi/h} = 180 \times \dfrac{5280}{3600} \text{ ft/s} = 264 \text{ ft/s}$. Then $a(t) = 2.4 \text{ ft/s}^2 \quad \Rightarrow \quad v(t) = 2.4t + C$, but

$v(0) = 0 \quad \Rightarrow \quad C = 0$. Now $2.4t = 264$ when $t = \dfrac{264}{2.4} = 110$ s, so it takes 110 s to reach 264 ft/s. Therefore, taking $s(0) = 0$, we have $s(t) = 1.2t^2$, $0 \le t \le 110$. So $s(110) = 14{,}520$ ft. 20 minutes $= 20(60) = 1200$ s, so for $110 \le t \le 1310$ we have $v(t) = 264 \text{ ft/s} \quad \Rightarrow \quad s(1310) = 264(1200) + 14{,}520 = 331{,}320 \text{ ft} = 62.75 \text{ mi}$.

(b) As in part (a), the train accelerates for 110 s and travels 14,520 ft while doing so. Similarly, it decelerates for 110 s and travels 14,520 ft at the end of its trip. During the remaining $1200 - 2(110) = 980$ s it travels at 264 ft/s, so the distance traveled is $264 \cdot 980 = 258{,}720$ ft. Thus, the total distance is $14{,}520 + 258{,}720 + 14{,}520 = 287{,}760 \text{ ft} = 54.5 \text{ mi}$.

(c) $60 \text{ mi} = 60(5280) = 316{,}800$ ft. Subtract $2(14{,}520)$ to take care of the speeding up and slowing down, and we have 287,760 ft at 264 ft/s for a trip of $287{,}760/264 = 1090$ s at 180 mi/h. The total time is $1090 + 2(110) = 1310 \text{ s} = 21 \text{ min } 50 \text{ s} = 21.8\overline{3} \text{ min}$.

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(d) $37.5(60) = 2250$ s. Then $2250 - 2(110) = 2030$ s at maximum speed. $2030(264) + 2(14{,}520) = 564{,}960$ total feet or $564{,}960/5280 = 107$ mi.

4 Review

TRUE-FALSE QUIZ

1. False. For example, take $f(x) = x^3$, then $f'(x) = 3x^2$ and $f'(0) = 0$, but $f(0) = 0$ is not a maximum or minimum; $(0, 0)$ is an inflection point.

2. False. For example, $f(x) = |x|$ has an absolute minimum at 0, but $f'(0)$ does not exist.

3. False. For example, $f(x) = x$ is continuous on $(0, 1)$ but attains neither a maximum nor a minimum value on $(0, 1)$. Don't confuse this with f being continuous on the *closed* interval $[a, b]$, which would make the statement true.

4. True. By the Mean Value Theorem, $f'(c) = \dfrac{f(1) - f(-1)}{1 - (-1)} = \dfrac{0}{2} = 0$. Note that $|c| < 1 \quad \Leftrightarrow \quad c \in (-1, 1)$.

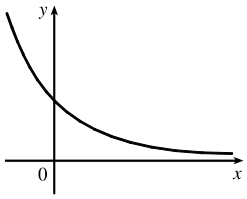
5. True. This is an example of part (b) of the Increasing/Decreasing Test.

6. False. For example, the curve $y = f(x) = 1$ has no inflection points but $f''(c) = 0$ for all c.

7. False. $f'(x) = g'(x) \quad \Rightarrow \quad f(x) = g(x) + C$. For example, if $f(x) = x + 2$ and $g(x) = x + 1$, then $f'(x) = g'(x) = 1$, but $f(x) \neq g(x)$.

8. False. Assume there is a function f such that $f(1) = -2$ and $f(3) = 0$. Then by the Mean Value Theorem there exists a number $c \in (1, 3)$ such that $f'(c) = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{0 - (-2)}{2} = 1$. But $f'(x) > 1$ for all x, a contradiction.

9. True. The graph of one such function is sketched.



10. False. At any point $(a, f(a))$, we know that $f'(a) < 0$. So since the tangent line at $(a, f(a))$ is not horizontal, it must cross the x-axis—at $x = b$, say. But since $f''(x) > 0$ for all x, the graph of f must lie above all of its tangents; in particular, $f(b) > 0$. But this is a contradiction, since we are given that $f(x) < 0$ for all x.

11. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $f(x_1) < f(x_2)$ and $g(x_1) < g(x_2)$ [since f and g are increasing on I], so $(f + g)(x_1) = f(x_1) + g(x_1) < f(x_2) + g(x_2) = (f + g)(x_2)$.

12. False. $f(x) = x$ and $g(x) = 2x$ are both increasing on $(0, 1)$, but $f(x) - g(x) = -x$ is not increasing on $(0, 1)$.

13. False. Take $f(x) = x$ and $g(x) = x - 1$. Then both f and g are increasing on $(0, 1)$. But $f(x)\, g(x) = x(x - 1)$ is not increasing on $(0, 1)$.

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14. True. Let $x_1 < x_2$ where $x_1, x_2 \in I$. Then $0 < f(x_1) < f(x_2)$ and $0 < g(x_1) < g(x_2)$ [since f and g are both positive and increasing]. Hence, $f(x_1)\,g(x_1) < f(x_2)\,g(x_1) < f(x_2)\,g(x_2)$. So fg is increasing on I.

15. True. Let $x_1, x_2 \in I$ and $x_1 < x_2$. Then $f(x_1) < f(x_2)$ [f is increasing] $\Rightarrow$ $\dfrac{1}{f(x_1)} > \dfrac{1}{f(x_2)}$ [f is positive] $\Rightarrow$ $g(x_1) > g(x_2) \Rightarrow g(x) = 1/f(x)$ is decreasing on I.

16. False. If f is even, then $f(x) = f(-x)$. Using the Chain Rule to differentiate this equation, we get $f'(x) = f'(-x)\,\dfrac{d}{dx}(-x) = -f'(-x)$. Thus, $f'(-x) = -f'(x)$, so f' is odd.

17. True. If f is periodic, then there is a number p such that $f(x+p) = f(p)$ for all x. Differentiating gives $f'(x) = f'(x+p)\cdot(x+p)' = f'(x+p)\cdot 1 = f'(x+p)$, so f' is periodic.

18. False. The most general antiderivative of $f(x) = x^{-2}$ is $F(x) = -1/x + C_1$ for $x < 0$ and $F(x) = -1/x + C_2$ for $x > 0$ [see Example 4.9.1(b)].

19. True. By the Mean Value Theorem, there exists a number c in $(0,1)$ such that $f(1) - f(0) = f'(c)(1-0) = f'(c)$. Since $f'(c)$ is nonzero, $f(1) - f(0) \neq 0$, so $f(1) \neq f(0)$.

20. False. Let $f(x) = 1 + \dfrac{1}{x}$ and $g(x) = x$. Then $\lim\limits_{x\to\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} g(x) = \infty$, but

$$\lim_{x\to\infty} [f(x)]^{g(x)} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e, \text{ not } 1.$$

21. False. $\lim\limits_{x\to 0} \dfrac{x}{e^x} = \dfrac{\lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} e^x} = \dfrac{0}{1} = 0$, not 1.

EXERCISES

1. $f(x) = x^3 - 9x^2 + 24x - 2$, $[0,5]$. $f'(x) = 3x^2 - 18x + 24 = 3(x^2 - 6x + 8) = 3(x-2)(x-4)$. $f'(x) = 0 \Leftrightarrow x = 2$ or $x = 4$. $f'(x) > 0$ for $0 < x < 2$, $f'(x) < 0$ for $2 < x < 4$, and $f'(x) > 0$ for $4 < x < 5$, so $f(2) = 18$ is a local maximum value and $f(4) = 14$ is a local minimum value. Checking the endpoints, we find $f(0) = -2$ and $f(5) = 18$. Thus, $f(0) = -2$ is the absolute minimum value and $f(2) = f(5) = 18$ is the absolute maximum value.

2. $f(x) = x\sqrt{1-x}$, $[-1,1]$. $f'(x) = x\cdot\frac{1}{2}(1-x)^{-1/2}(-1) + (1-x)^{1/2}(1) = (1-x)^{-1/2}\left[-\frac{1}{2}x + (1-x)\right] = \dfrac{1 - \frac{3}{2}x}{\sqrt{1-x}}$. $f'(x) = 0 \Rightarrow x = \frac{2}{3}$. $f'(x)$ does not exist $\Leftrightarrow x = 1$. $f'(x) > 0$ for $-1 < x < \frac{2}{3}$ and $f'(x) < 0$ for $\frac{2}{3} < x < 1$, so $f\left(\frac{2}{3}\right) = \frac{2}{3}\sqrt{\frac{1}{3}} = \frac{2}{9}\sqrt{3}$ [≈ 0.38] is a local maximum value. Checking the endpoints, we find $f(-1) = -\sqrt{2}$ and $f(1) = 0$. Thus, $f(-1) = -\sqrt{2}$ is the absolute minimum value and $f\left(\frac{2}{3}\right) = \frac{2}{9}\sqrt{3}$ is the absolute maximum value.

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3. $f(x)=\dfrac{3x-4}{x^2+1},\ [-2,2]$. $f'(x)=\dfrac{(x^2+1)(3)-(3x-4)(2x)}{(x^2+1)^2}=\dfrac{-(3x^2-8x-3)}{(x^2+1)^2}=\dfrac{-(3x+1)(x-3)}{(x^2+1)^2}$.

$f'(x)=0 \ \Rightarrow\ x=-\frac{1}{3}$ or $x=3$, but 3 is not in the interval. $f'(x)>0$ for $-\frac{1}{3}<x<2$ and $f'(x)<0$ for $-2<x<-\frac{1}{3}$, so $f\left(-\frac{1}{3}\right)=\frac{-5}{10/9}=-\frac{9}{2}$ is a local minimum value. Checking the endpoints, we find $f(-2)=-2$ and $f(2)=\frac{2}{5}$. Thus, $f\left(-\frac{1}{3}\right)=-\frac{9}{2}$ is the absolute minimum value and $f(2)=\frac{2}{5}$ is the absolute maximum value.

4. $f(x)=\sqrt{x^2+x+1},\ [-2,1]$. $f'(x)=\frac{1}{2}(x^2+x+1)^{-1/2}(2x+1)=\dfrac{2x+1}{2\sqrt{x^2+x+1}}$. $f'(x)=0 \ \Rightarrow\ x=-\dfrac{1}{2}$.

$f'(x)>0$ for $-\frac{1}{2}<x<1$ and $f'(x)<0$ for $-2<x<-\frac{1}{2}$, so $f\left(-\frac{1}{2}\right)=\sqrt{3}/2$ is a local minimum value. Checking the endpoints, we find $f(-2)=f(1)=\sqrt{3}$. Thus, $f\left(-\frac{1}{2}\right)=\sqrt{3}/2$ is the absolute minimum value and $f(-2)=f(1)=\sqrt{3}$ is the absolute maximum value.

5. $f(x)=x+2\cos x,\ [-\pi,\pi]$. $f'(x)=1-2\sin x$. $f'(x)=0 \ \Rightarrow\ \sin x=\frac{1}{2} \ \Rightarrow\ x=\frac{\pi}{6},\frac{5\pi}{6}$. $f'(x)>0$ for $\left(-\pi,\frac{\pi}{6}\right)$ and $\left(\frac{5\pi}{6},\pi\right)$, and $f'(x)<0$ for $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$, so $f\left(\frac{\pi}{6}\right)=\frac{\pi}{6}+\sqrt{3}\approx 2.26$ is a local maximum value and $f\left(\frac{5\pi}{6}\right)=\frac{5\pi}{6}-\sqrt{3}\approx 0.89$ is a local minimum value. Checking the endpoints, we find $f(-\pi)=-\pi-2\approx -5.14$ and $f(\pi)=\pi-2\approx 1.14$. Thus, $f(-\pi)=-\pi-2$ is the absolute minimum value and $f\left(\frac{\pi}{6}\right)=\frac{\pi}{6}+\sqrt{3}$ is the absolute maximum value.

6. $f(x)=x^2e^{-x}$, $[-1,3]$. $f'(x)=x^2(-e^{-x})+e^{-x}(2x)=xe^{-x}(-x+2)$. $f'(x)=0 \ \Rightarrow\ x=0$ or $x=2$.

$f'(x)>0$ for $0<x<2$ and $f'(x)<0$ for $-1<x<0$ and $2<x<3$, so $f(0)=0$ is a local minimum value and $f(2)=4e^{-2}\approx 0.54$ is a local maximum value. Checking the endpoints, we find $f(-1)=e\approx 2.72$ and $f(3)=9e^{-3}\approx 0.45$. Thus, $f(0)=0$ is the absolute minimum value and $f(-1)=e$ is the absolute maximum value.

7. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^x-1}{\tan x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{e^x}{\sec^2 x}=\frac{1}{1}=1$

8. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{\tan 4x}{x+\sin 2x}\overset{\text{H}}{=}\lim_{x\to 0}\frac{4\sec^2 4x}{1+2\cos 2x}=\frac{4(1)}{1+2(1)}=\frac{4}{3}$

9. This limit has the form $\frac{0}{0}$. $\displaystyle\lim_{x\to 0}\frac{e^{2x}-e^{-2x}}{\ln(x+1)}\overset{\text{H}}{=}\lim_{x\to 0}\frac{2e^{2x}+2e^{-2x}}{1/(x+1)}=\frac{2+2}{1}=4$

10. This limit has the form $\frac{\infty}{\infty}$. $\displaystyle\lim_{x\to\infty}\frac{e^{2x}-e^{-2x}}{\ln(x+1)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2e^{2x}+2e^{-2x}}{1/(x+1)}=\lim_{x\to\infty}2(x+1)(e^{2x}+e^{-2x})=\infty$

since $2(x+1)\to\infty$ and $(e^{2x}+e^{-2x})\to\infty$ as $x\to\infty$.

11. This limit has the form $\infty\cdot 0$.

$$\lim_{x\to-\infty}(x^2-x^3)e^{2x}=\lim_{x\to-\infty}\frac{x^2-x^3}{e^{-2x}}\ \left[\tfrac{\infty}{\infty}\text{ form}\right]\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{2x-3x^2}{-2e^{-2x}}\ \left[\tfrac{\infty}{\infty}\text{ form}\right]$$
$$\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{2-6x}{4e^{-2x}}\ \left[\tfrac{\infty}{\infty}\text{ form}\right]\overset{\text{H}}{=}\lim_{x\to-\infty}\frac{-6}{-8e^{-2x}}=0$$

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12. This limit has the form $0 \cdot \infty$. $\lim\limits_{x\to\pi^-}(x-\pi)\csc x = \lim\limits_{x\to\pi^-}\dfrac{x-\pi}{\sin x}$ $\left[\frac{0}{0}\text{ form}\right]$ $\overset{\text{H}}{=}$ $\lim\limits_{x\to\pi^-}\dfrac{1}{\cos x} = \dfrac{1}{-1} = -1$

13. This limit has the form $\infty - \infty$.

$$\lim_{x\to1^+}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right) = \lim_{x\to1^+}\left(\frac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim_{x\to1^+}\frac{x\cdot(1/x)+\ln x - 1}{(x-1)\cdot(1/x)+\ln x} = \lim_{x\to1^+}\frac{\ln x}{1-1/x+\ln x}$$

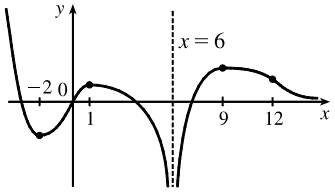
$$\overset{\text{H}}{=} \lim_{x\to1^+}\frac{1/x}{1/x^2+1/x} = \frac{1}{1+1} = \frac{1}{2}$$

14. $y = (\tan x)^{\cos x} \quad\Rightarrow\quad \ln y = \cos x \, \ln\tan x$, so

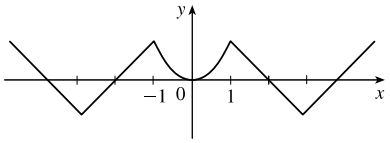
$$\lim_{x\to(\pi/2)^-}\ln y = \lim_{x\to(\pi/2)^-}\frac{\ln\tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-}\frac{(1/\tan x)\sec^2 x}{\sec x\,\tan x} = \lim_{x\to(\pi/2)^-}\frac{\sec x}{\tan^2 x} = \lim_{x\to(\pi/2)^-}\frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0,$$

so $\lim\limits_{x\to(\pi/2)^-}(\tan x)^{\cos x} = \lim\limits_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1$.

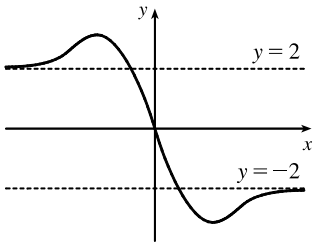
15. $f(0) = 0$, $f'(-2) = f'(1) = f'(9) = 0$, $\lim\limits_{x\to\infty} f(x) = 0$, $\lim\limits_{x\to6} f(x) = -\infty$,
$f'(x) < 0$ on $(-\infty,-2)$, $(1,6)$, and $(9,\infty)$, $f'(x) > 0$ on $(-2,1)$ and $(6,9)$,
$f''(x) > 0$ on $(-\infty,0)$ and $(12,\infty)$, $f''(x) < 0$ on $(0,6)$ and $(6,12)$



16. For $0 < x < 1$, $f'(x) = 2x$, so $f(x) = x^2 + C$. Since $f(0) = 0$,
$f(x) = x^2$ on $[0,1]$. For $1 < x < 3$, $f'(x) = -1$, so $f(x) = -x + D$.
$1 = f(1) = -1 + D \quad\Rightarrow\quad D = 2$, so $f(x) = 2 - x$. For $x > 3$, $f'(x) = 1$,
so $f(x) = x + E$. $-1 = f(3) = 3 + E \quad\Rightarrow\quad E = -4$, so $f(x) = x - 4$.
Since f is even, its graph is symmetric about the y-axis.



17. f is odd, $f'(x) < 0$ for $0 < x < 2$, $\quad f'(x) > 0$ for $x > 2$,
$f''(x) > 0$ for $0 < x < 3$, $\quad f''(x) < 0$ for $x > 3$,
$\lim_{x\to\infty} f(x) = -2$



18. (a) Since $f'(x) > 0$ on $(1,4)$ and $(4,7)$, f is increasing on these intervals. Since $f'(x) < 0$ on $(0,1)$ and $(7,8)$, f is decreasing on these intervals.

(b) Since $f'(7) = 0$ and f' changes from positive to negative there, f changes from increasing to decreasing and has a local maximum at $x = 7$. Since $f'(x) = 0$ at $x = 1$ and f' changes from negative to positive there, f changes from decreasing to increasing and has a local minimum at $x = 1$.

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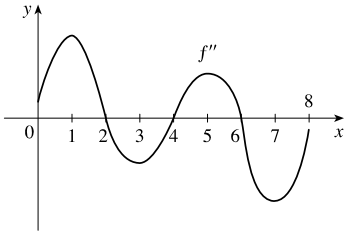
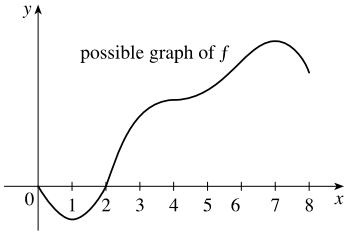
(c)



(d)

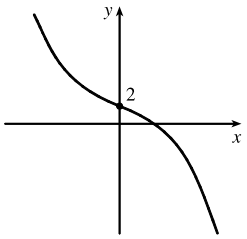


19. $y = f(x) = 2 - 2x - x^3$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$. The x-intercept (approximately 0.770917) can be found using Newton's Method. **C.** No symmetry **D.** No asymptote

E. $f'(x) = -2 - 3x^2 = -(3x^2 + 2) < 0$, so f is decreasing on $\mathbb{R}$.

F. No extreme value **G.** $f''(x) = -6x < 0$ on $(0, \infty)$ and $f''(x) > 0$ on $(-\infty, 0)$, so f is CD on $(0, \infty)$ and CU on $(-\infty, 0)$. There is an IP at $(0, 2)$.

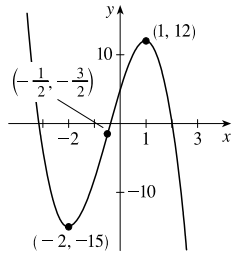
H.



20. $y = f(x) = -2x^3 - 3x^2 + 12x + 5$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 5$; x-intercept: $f(x) = 0 \Leftrightarrow x \approx -3.15, -0.39, 2.04$ **C.** No symmetry **D.** No asymptote

E. $f'(x) = -6x^2 - 6x + 12 = -6(x^2 + x - 2) = -6(x + 2)(x - 1)$. $f'(x) > 0$ for $-2 < x < 1$, so f is increasing on $(-2, 1)$ and decreasing on $(-\infty, -2)$ and $(1, \infty)$. **F.** Local minimum value $f(-2) = -15$, local maximum value $f(1) = 12$ **G.** $f''(x) = -12x - 6 = -12\left(x + \frac{1}{2}\right)$. $f''(x) > 0$ for $x < -\frac{1}{2}$, so f is CU on $\left(-\infty, -\frac{1}{2}\right)$ and CD on $\left(-\frac{1}{2}, \infty\right)$. There is an IP at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$.

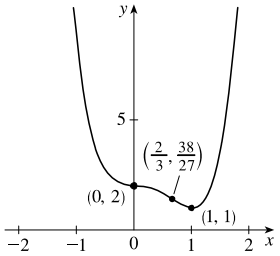
H.



21. $y = f(x) = 3x^4 - 4x^3 + 2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptote

E. $f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1)$. $f'(x) > 0$ for $x > 1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, 1)$. **F.** $f'(x)$ does not change sign at $x = 0$, so there is no local extremum there. $f(1) = 1$ is a local minimum value. **G.** $f''(x) = 36x^2 - 24x = 12x(3x - 2)$. $f''(x) < 0$ for $0 < x < \frac{2}{3}$, so f is CD on $\left(0, \frac{2}{3}\right)$ and f is CU on $(-\infty, 0)$ and $\left(\frac{2}{3}, \infty\right)$. There are inflection points at $(0, 2)$ and $\left(\frac{2}{3}, \frac{38}{27}\right)$.

H.



22. $y = f(x) = \dfrac{x}{1 - x^2}$ **A.** $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercept: 0

C. $f(-x) = -f(x)$, so f is odd and the graph is symmetric about the origin. **D.** $\lim\limits_{x \to \pm\infty} \dfrac{x}{1 - x^2} = 0$, so $y = 0$ is a HA.

$\lim\limits_{x \to -1^-} \dfrac{x}{1 - x^2} = \infty$ and $\lim\limits_{x \to -1^+} \dfrac{x}{1 - x^2} = -\infty$, so $x = -1$ is a VA. Similarly, $\lim\limits_{x \to 1^-} \dfrac{x}{1 - x^2} = \infty$ and

$\lim\limits_{x \to 1^+} \dfrac{x}{1 - x^2} = -\infty$, so $x = 1$ is a VA. **E.** $f'(x) = \dfrac{(1 - x^2)(1) - x(-2x)}{(1 - x^2)^2} = \dfrac{1 + x^2}{(1 - x^2)^2} > 0$ for $x \neq \pm 1$, so f is

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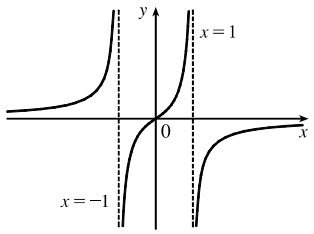
increasing on $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$. **F.** No local extrema

G. $f''(x)=\dfrac{(1-x^2)^2(2x)-(1+x^2)2(1-x^2)(-2x)}{[(1-x^2)^2]^2}$

$=\dfrac{2x(1-x^2)[(1-x^2)+2(1+x^2)]}{(1-x^2)^4}=\dfrac{2x(3+x^2)}{(1-x^2)^3}$

$f''(x)>0$ for $x<-1$ and $0<x<1$, and $f''(x)<0$ for $-1<x<0$ and $x>1$, so f is CU on $(-\infty,-1)$ and $(0,1)$, and f is CD on $(-1,0)$ and $(1,\infty)$. $(0,0)$ is an IP.

H.



23. $y=f(x)=\dfrac{1}{x(x-3)^2}$ **A.** $D=\{x\mid x\neq 0,3\}=(-\infty,0)\cup(0,3)\cup(3,\infty)$ **B.** No intercepts. **C.** No symmetry.

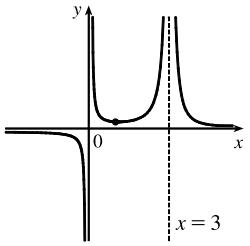
D. $\lim\limits_{x\to\pm\infty}\dfrac{1}{x(x-3)^2}=0$, so $y=0$ is a HA. $\lim\limits_{x\to 0^+}\dfrac{1}{x(x-3)^2}=\infty$, $\lim\limits_{x\to 0^-}\dfrac{1}{x(x-3)^2}=-\infty$, $\lim\limits_{x\to 3}\dfrac{1}{x(x-3)^2}=\infty$, so $x=0$ and $x=3$ are VA. **E.** $f'(x)=-\dfrac{(x-3)^2+2x(x-3)}{x^2(x-3)^4}=\dfrac{3(1-x)}{x^2(x-3)^3}\quad\Rightarrow\quad f'(x)>0\quad\Leftrightarrow\quad 1<x<3$, so f is increasing on $(1,3)$ and decreasing on $(-\infty,0)$, $(0,1)$, and $(3,\infty)$.

F. Local minimum value $f(1)=\frac{1}{4}$ **G.** $f''(x)=\dfrac{6(2x^2-4x+3)}{x^3(x-3)^4}$.

Note that $2x^2-4x+3>0$ for all x since it has negative discriminant.

So $f''(x)>0\quad\Leftrightarrow\quad x>0\quad\Rightarrow\quad f$ is CU on $(0,3)$ and $(3,\infty)$ and CD on $(-\infty,0)$. No IP

H.



24. $y=f(x)=\dfrac{1}{x^2}-\dfrac{1}{(x-2)^2}$ **A.** $D=\{x\mid x\neq 0,2\}$ **B.** y-intercept: none; x-intercept: $f(x)=0\quad\Rightarrow$

$\dfrac{1}{x^2}=\dfrac{1}{(x-2)^2}\quad\Leftrightarrow\quad(x-2)^2=x^2\quad\Leftrightarrow\quad x^2-4x+4=x^2\quad\Leftrightarrow\quad 4x=4\quad\Leftrightarrow\quad x=1$ **C.** No symmetry

D. $\lim\limits_{x\to 0}f(x)=\infty$ and $\lim\limits_{x\to 2}f(x)=-\infty$, so $x=0$ and $x=2$ are VA; $\lim\limits_{x\to\pm\infty}f(x)=0$, so $y=0$ is a HA

E. $f'(x)=-\dfrac{2}{x^3}+\dfrac{2}{(x-2)^3}>0\quad\Rightarrow\quad\dfrac{-(x-2)^3+x^3}{x^3(x-2)^3}>0\quad\Leftrightarrow\quad\dfrac{-x^3+6x^2-12x+8+x^3}{x^3(x-2)^3}>0\quad\Leftrightarrow$

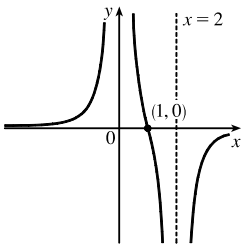
$\dfrac{2(3x^2-6x+4)}{x^3(x-2)^3}>0$. The numerator is positive (the discriminant of the quadratic is negative), so $f'(x)>0$ if $x<0$ or $x>2$, and hence, f is increasing on $(-\infty,0)$ and $(2,\infty)$ and decreasing on $(0,2)$.

F. No local extreme values **G.** $f''(x)=\dfrac{6}{x^4}-\dfrac{6}{(x-2)^4}>0\quad\Rightarrow$

$\dfrac{(x-2)^4-x^4}{x^4(x-2)^4}>0\quad\Leftrightarrow\quad\dfrac{x^4-8x^3+24x^2-32x+16-x^4}{x^4(x-2)^4}>0\quad\Leftrightarrow$

$\dfrac{-8(x^3-3x^2+4x-2)}{x^4(x-2)^4}>0\quad\Leftrightarrow\quad\dfrac{-8(x-1)(x^2-2x+2)}{x^4(x-2)^4}>0$. So f'' is positive for $x<1$ $[x\neq 0]$ and negative for $x>1$ $[x\neq 2]$. Thus, f is CU on $(-\infty,0)$ and $(0,1)$ and f is CD on $(1,2)$ and $(2,\infty)$. IP at $(1,0)$

H.



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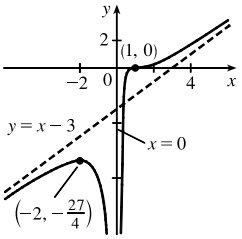
25. $y = f(x) = \dfrac{(x-1)^3}{x^2} = \dfrac{x^3 - 3x^2 + 3x - 1}{x^2} = x - 3 + \dfrac{3x-1}{x^2}$ **A.** $D = \{x \mid x \neq 0\} = (-\infty, 0) \cup (0, \infty)$

B. y-intercept: none; x-intercept: $f(x) = 0 \;\Leftrightarrow\; x = 1$ **C.** No symmetry **D.** $\lim\limits_{x \to 0^-} \dfrac{(x-1)^3}{x^2} = -\infty$ and

$\lim\limits_{x \to 0^+} f(x) = -\infty$, so $x = 0$ is a VA. $f(x) - (x-3) = \dfrac{3x-1}{x^2} \to 0$ as $x \to \pm\infty$, so $y = x - 3$ is a SA.

E. $f'(x) = \dfrac{x^2 \cdot 3(x-1)^2 - (x-1)^3(2x)}{(x^2)^2} = \dfrac{x(x-1)^2[3x - 2(x-1)]}{x^4} = \dfrac{(x-1)^2(x+2)}{x^3}$. $f'(x) < 0$ for $-2 < x < 0$,

so f is increasing on $(-\infty, -2)$, decreasing on $(-2, 0)$, and increasing on $(0, \infty)$. **H.**



F. Local maximum value $f(-2) = -\frac{27}{4}$ **G.** $f(x) = x - 3 + \dfrac{3}{x} - \dfrac{1}{x^2} \;\Rightarrow$

$f'(x) = 1 - \dfrac{3}{x^2} + \dfrac{2}{x^3} \;\Rightarrow\; f''(x) = \dfrac{6}{x^3} - \dfrac{6}{x^4} = \dfrac{6x-6}{x^4} = \dfrac{6(x-1)}{x^4}$.

$f''(x) > 0$ for $x > 1$, so f is CD on $(-\infty, 0)$ and $(0, 1)$, and f is CU on $(1, \infty)$.

There is an inflection point at $(1, 0)$.

26. $y = f(x) = \sqrt{1-x} + \sqrt{1+x}$ **A.** $1 - x \geq 0$ and $1 + x \geq 0 \;\Rightarrow\; x \leq 1$ and $x \geq -1$, so $D = [-1, 1]$.

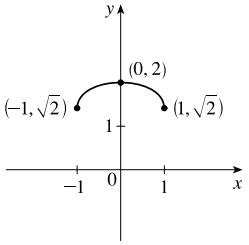
B. y-intercept: $f(0) = 1 + 1 = 2$; no x-intercept because $f(x) > 0$ for all x.

C. $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** No asymptote

E. $f'(x) = \frac{1}{2}(1-x)^{-1/2}(-1) + \frac{1}{2}(1+x)^{-1/2} = \dfrac{-1}{2\sqrt{1-x}} + \dfrac{1}{2\sqrt{1+x}} = \dfrac{-\sqrt{1+x} + \sqrt{1-x}}{2\sqrt{1-x}\sqrt{1+x}} > 0 \;\Rightarrow$

$-\sqrt{1+x} + \sqrt{1-x} > 0 \;\Rightarrow\; \sqrt{1-x} > \sqrt{1+x} \;\Rightarrow\; 1 - x > 1 + x \;\Rightarrow\; -2x > 0 \;\Rightarrow\; x < 0$, so $f'(x) > 0$ for

$-1 < x < 0$ and $f'(x) < 0$ for $0 < x < 1$. Thus, f is increasing on $(-1, 0)$ **H.**



and decreasing on $(0, 1)$. **F.** Local maximum value $f(0) = 2$

G. $f''(x) = -\frac{1}{2}\left(-\frac{1}{2}\right)(1-x)^{-3/2}(-1) + \frac{1}{2}\left(-\frac{1}{2}\right)(1+x)^{-3/2}$

$= \dfrac{-1}{4(1-x)^{3/2}} + \dfrac{-1}{4(1+x)^{3/2}} < 0$

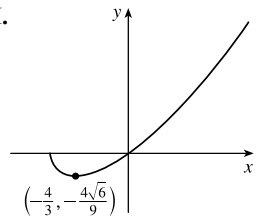
for all x in the domain, so f is CD on $(-1, 1)$. No IP

27. $y = f(x) = x\sqrt{2+x}$ **A.** $D = [-2, \infty)$ **B.** y-intercept: $f(0) = 0$; x-intercepts: -2 and 0 **C.** No symmetry

D. No asymptote **E.** $f'(x) = \dfrac{x}{2\sqrt{2+x}} + \sqrt{2+x} = \dfrac{1}{2\sqrt{2+x}}\,[x + 2(2+x)] = \dfrac{3x+4}{2\sqrt{2+x}} = 0$ when $x = -\frac{4}{3}$, so f is

decreasing on $\left(-2, -\frac{4}{3}\right)$ and increasing on $\left(-\frac{4}{3}, \infty\right)$. **F.** Local minimum value $f\left(-\frac{4}{3}\right) = -\frac{4}{3}\sqrt{\frac{2}{3}} = -\frac{4\sqrt{6}}{9} \approx -1.09$,

no local maximum **H.**



G. $f''(x) = \dfrac{2\sqrt{2+x} \cdot 3 - (3x+4)\dfrac{1}{\sqrt{2+x}}}{4(2+x)} = \dfrac{6(2+x) - (3x+4)}{4(2+x)^{3/2}}$

$= \dfrac{3x+8}{4(2+x)^{3/2}}$

$f''(x) > 0$ for $x > -2$, so f is CU on $(-2, \infty)$. No IP

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28. $y = f(x) = x^{2/3}(x-3)^2$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Leftrightarrow x = 0, 3$

C. No symmetry **D.** No asymptote

E. $f'(x) = x^{2/3} \cdot 2(x-3) + (x-3)^2 \cdot \frac{2}{3}x^{-1/3} = \frac{2}{3}x^{-1/3}(x-3)[3x + (x-3)] = \frac{2}{3}x^{-1/3}(x-3)(4x-3)$.

$f'(x) > 0 \Leftrightarrow 0 < x < \frac{3}{4}$ or $x > 3$, so f is decreasing on $(-\infty, 0)$, increasing on $\left(0, \frac{3}{4}\right)$, decreasing on $\left(\frac{3}{4}, 3\right)$, and increasing on $(3, \infty)$. **F.** Local minimum value $f(0) = f(3) = 0$; local maximum value

$f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^{2/3}\left(-\frac{9}{4}\right)^2 = \frac{81}{16}\sqrt[3]{\frac{9}{16}} = \frac{81}{32}\sqrt[3]{\frac{9}{2}}$ $[\approx 4.18]$

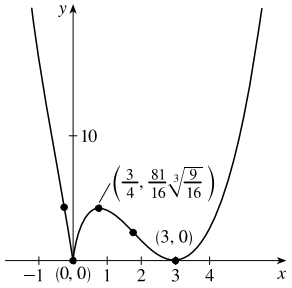
G. $f'(x) = \left(\frac{2}{3}x^{-1/3}\right)(4x^2 - 15x + 9) \Rightarrow$

$$\begin{aligned} f''(x) &= \left(\tfrac{2}{3}x^{-1/3}\right)(8x-15) + (4x^2-15x+9)\left(-\tfrac{2}{9}x^{-4/3}\right) \\ &= \tfrac{2}{9}x^{-4/3}[3x(8x-15) - (4x^2-15x+9)] \\ &= \tfrac{2}{9}x^{-4/3}(20x^2 - 30x - 9) \end{aligned}$$

$f''(x) = 0 \Leftrightarrow x \approx -0.26$ or 1.76. $f''(x)$ does not exist at $x = 0$.

f is CU on $(-\infty, -0.26)$, CD on $(-0.26, 0)$, CD on $(0, 1.76)$, and CU on $(1.76, \infty)$. There are inflection points at $(-0.26, 4.28)$ and $(1.76, 2.25)$.

H.



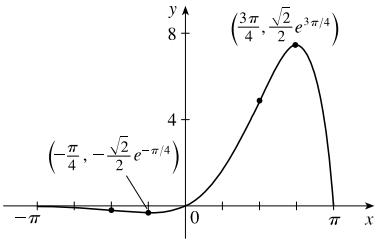
29. $y = f(x) = e^x \sin x$, $-\pi \le x \le \pi$ **A.** $D = [-\pi, \pi]$ **B.** y-intercept: $f(0) = 0$; $f(x) = 0 \Leftrightarrow \sin x = 0 \Rightarrow x = -\pi, 0, \pi$. **C.** No symmetry **D.** No asymptote **E.** $f'(x) = e^x \cos x + \sin x \cdot e^x = e^x(\cos x + \sin x)$.

$f'(x) = 0 \Leftrightarrow -\cos x = \sin x \Leftrightarrow -1 = \tan x \Rightarrow x = -\frac{\pi}{4}, \frac{3\pi}{4}$. $f'(x) > 0$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$ and $f'(x) < 0$ for $-\pi < x < -\frac{\pi}{4}$ and $\frac{3\pi}{4} < x < \pi$, so f is increasing on $\left(-\frac{\pi}{4}, \frac{3\pi}{4}\right)$ and f is decreasing on $\left(-\pi, -\frac{\pi}{4}\right)$ and $\left(\frac{3\pi}{4}, \pi\right)$.

F. Local minimum value $f\left(-\frac{\pi}{4}\right) = (-\sqrt{2}/2)e^{-\pi/4} \approx -0.32$ and local maximum value $f\left(\frac{3\pi}{4}\right) = \left(\sqrt{2}/2\right)e^{3\pi/4} \approx 7.46$

G. $f''(x) = e^x(-\sin x + \cos x) + (\cos x + \sin x)e^x = e^x(2\cos x) > 0 \Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$ and $f''(x) < 0 \Rightarrow -\pi < x < -\frac{\pi}{2}$ and $\frac{\pi}{2} < x < \pi$, so f is CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, and f is CD on $\left(-\pi, -\frac{\pi}{2}\right)$ and $\left(\frac{\pi}{2}, \pi\right)$. There are inflection points at $\left(-\frac{\pi}{2}, -e^{-\pi/2}\right)$ and $\left(\frac{\pi}{2}, e^{\pi/2}\right)$.

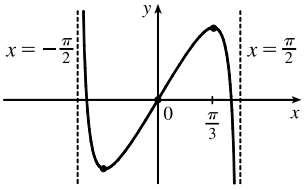
H.



30. $y = f(x) = 4x - \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ **A.** $D = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. **B.** y-intercept $= f(0) = 0$ **C.** $f(-x) = -f(x)$, so the curve is symmetric about $(0, 0)$. **D.** $\lim_{x \to \pi/2^-}(4x - \tan x) = -\infty$, $\lim_{x \to -\pi/2^+}(4x - \tan x) = \infty$, so $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VA. **E.** $f'(x) = 4 - \sec^2 x > 0 \Leftrightarrow \sec x < 2 \Leftrightarrow \cos x > \frac{1}{2} \Leftrightarrow -\frac{\pi}{3} < x < \frac{\pi}{3}$, so f is increasing on $\left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ and decreasing on $\left(-\frac{\pi}{2}, -\frac{\pi}{3}\right)$ and $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$. **F.** $f\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} - \sqrt{3}$ is a local maximum value, $f\left(-\frac{\pi}{3}\right) = \sqrt{3} - \frac{4\pi}{3}$ is a local minimum value.

G. $f''(x) = -2\sec^2 x \tan x > 0 \Leftrightarrow \tan x < 0 \Leftrightarrow -\frac{\pi}{2} < x < 0$, so f is CU on $\left(-\frac{\pi}{2}, 0\right)$ and CD on $\left(0, \frac{\pi}{2}\right)$. IP at $(0, 0)$

H.



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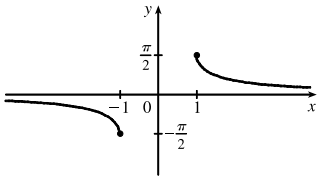
31. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept

C. $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x \to \pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

H.

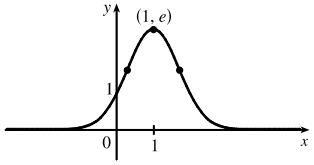


G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2 - 1)}{(x^4 - x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

32. $y = f(x) = e^{2x - x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim_{x \to \pm\infty} e^{2x - x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x - x^2} \Rightarrow f'(x) = 2(1 - x)e^{2x - x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2(2x^2 - 4x + 1)e^{2x - x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$.
$f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.

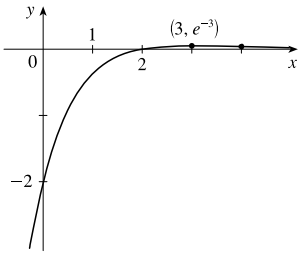


33. $y = f(x) = (x - 2)e^{-x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = -2$; x-intercept: $f(x) = 0 \Leftrightarrow x = 2$

C. No symmetry **D.** $\lim_{x \to \infty} \dfrac{x - 2}{e^x} \overset{\text{H}}{=} \lim_{x \to \infty} \dfrac{1}{e^x} = 0$, so $y = 0$ is a HA. No VA

E. $f'(x) = (x - 2)(-e^{-x}) + e^{-x}(1) = e^{-x}[-(x - 2) + 1] = (3 - x)e^{-x}$.
$f'(x) > 0$ for $x < 3$, so f is increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$.

H.



F. Local maximum value $f(3) = e^{-3}$, no local minimum value

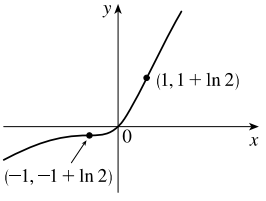
G. $f''(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = e^{-x}[-(3 - x) + (-1)]$
$= (x - 4)e^{-x} > 0$
for $x > 4$, so f is CU on $(4, \infty)$ and CD on $(-\infty, 4)$. IP at $(4, 2e^{-4})$

34. $y = f(x) = x + \ln(x^2 + 1)$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0 + \ln 1 = 0$; x-intercept: $f(x) = 0 \Leftrightarrow \ln(x^2 + 1) = -x \Leftrightarrow x^2 + 1 = e^{-x} \Rightarrow x = 0$ since the graphs of $y = x^2 + 1$ and $y = e^{-x}$ intersect only at $x = 0$.

C. No symmetry **D.** No asymptote **E.** $f'(x) = 1 + \dfrac{2x}{x^2 + 1} = \dfrac{x^2 + 2x + 1}{x^2 + 1} = \dfrac{(x + 1)^2}{x^2 + 1}$. $f'(x) > 0$ if $x \ne -1$ and f is increasing on $\mathbb{R}$. **F.** No local extreme values

G. $f''(x) = \dfrac{(x^2 + 1)2 - 2x(2x)}{(x^2 + 1)^2} = \dfrac{2[(x^2 + 1) - 2x^2]}{(x^2 + 1)^2} = \dfrac{2(1 - x^2)}{(x^2 + 1)^2}$.
$f''(x) > 0 \Leftrightarrow -1 < x < 1$ and $f''(x) < 0 \Leftrightarrow x < -1$ or $x > 1$, so f is CU on $(-1, 1)$ and f is CD on $(-\infty, -1)$ and $(1, \infty)$. IP at $(-1, -1 + \ln 2)$ and $(1, 1 + \ln 2)$

H.



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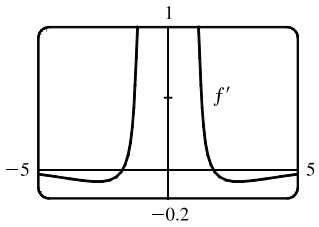
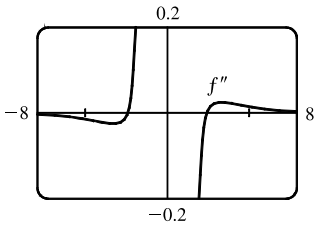
35. $f(x)=\dfrac{x^2-1}{x^3} \;\Rightarrow\; f'(x)=\dfrac{x^3(2x)-(x^2-1)3x^2}{x^6}=\dfrac{3-x^2}{x^4} \;\Rightarrow$

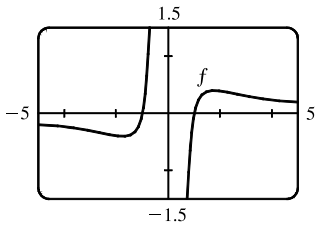
$f''(x)=\dfrac{x^4(-2x)-(3-x^2)4x^3}{x^8}=\dfrac{2x^2-12}{x^5}$



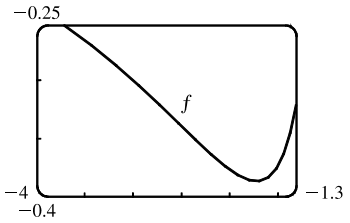
Estimates: From the graphs of f' and f'', it appears that f is increasing on $(-1.73, 0)$ and $(0, 1.73)$ and decreasing on $(-\infty, -1.73)$ and $(1.73, \infty)$; f has a local maximum of about $f(1.73) = 0.38$ and a local minimum of about $f(-1.7) = -0.38$; f is CU on $(-2.45, 0)$ and $(2.45, \infty)$, and CD on $(-\infty, -2.45)$ and $(0, 2.45)$; and f has inflection points at about $(-2.45, -0.34)$ and $(2.45, 0.34)$.



Exact: Now $f'(x)=\dfrac{3-x^2}{x^4}$ is positive for $0 < x^2 < 3$, that is, f is increasing on $\left(-\sqrt{3}, 0\right)$ and $\left(0, \sqrt{3}\right)$; and $f'(x)$ is negative (and so f is decreasing) on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$. $f'(x) = 0$ when $x = \pm\sqrt{3}$.

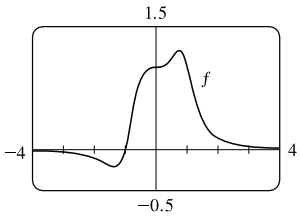


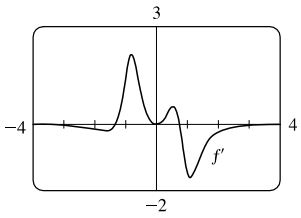
f' goes from positive to negative at $x = \sqrt{3}$, so f has a local maximum of $f\left(\sqrt{3}\right)=\frac{\left(\sqrt{3}\right)^2-1}{\left(\sqrt{3}\right)^3}=\frac{2\sqrt{3}}{9}$; and since f is odd, we know that maxima on the interval $(0, \infty)$ correspond to minima on $(-\infty, 0)$, so f has a local minimum of $f\left(-\sqrt{3}\right)=-\frac{2\sqrt{3}}{9}$. Also, $f''(x)=\dfrac{2x^2-12}{x^5}$ is positive (so f is CU) on $\left(-\sqrt{6}, 0\right)$ and $\left(\sqrt{6}, \infty\right)$, and negative (so f is CD) on $\left(-\infty, -\sqrt{6}\right)$ and $\left(0, \sqrt{6}\right)$. There are IP at $\left(\sqrt{6}, \frac{5\sqrt{6}}{36}\right)$ and $\left(-\sqrt{6}, -\frac{5\sqrt{6}}{36}\right)$.

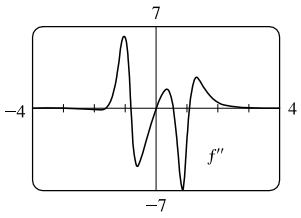


36. $f(x)=\dfrac{x^3+1}{x^6+1} \;\Rightarrow\; f'(x)=-\dfrac{3x^2(x^6+2x^3-1)}{(x^6+1)^2} \;\Rightarrow\; f''(x)=\dfrac{6x(2x^{12}+7x^9-9x^6-5x^3+1)}{(x^6+1)^3}.$

$f(x) = 0 \;\Leftrightarrow\; x = -1$. $f'(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx -1.34, 0.75$. $f''(x) = 0 \;\Leftrightarrow\; x = 0$ or $x \approx -1.64, -0.82, 0.54, 1.09$. From the graphs of f and f', it appears that f is decreasing on $(-\infty, -1.34)$, increasing on $(-1.34, 0.75)$, and decreasing on $(0.75, \infty)$. f has a local minimum value of $f(-1.34) \approx -0.21$ and a local maximum value of $f(0.75) \approx 1.21$. From the graphs of f and f'', it appears that f is CD on $(-\infty, -1.64)$, CU on $(-1.64, -0.82)$, CD on $(-0.82, 0)$, CU on $(0, 0.54)$, CD on $(0.54, 1.09)$ and CU on $(1.09, \infty)$. There are inflection points at about $(-1.64, -0.17)$, $(-0.82, 0.34)$, $(0.54, 1.13)$, $(1.09, 0.86)$, and at $(0, 1)$.







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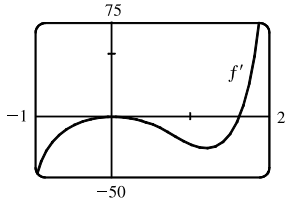
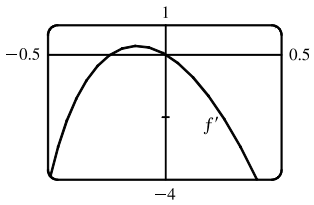
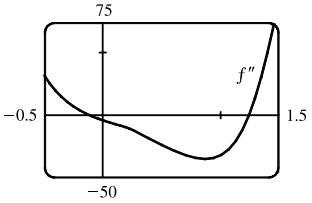
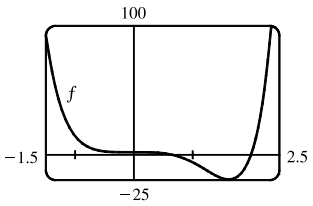
37. $f(x) = 3x^6 - 5x^5 + x^4 - 5x^3 - 2x^2 + 2 \;\Rightarrow\; f'(x) = 18x^5 - 25x^4 + 4x^3 - 15x^2 - 4x \;\Rightarrow$
$f''(x) = 90x^4 - 100x^3 + 12x^2 - 30x - 4$

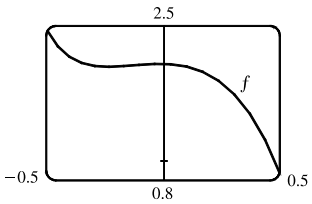




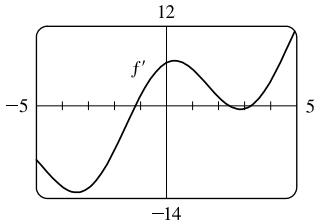


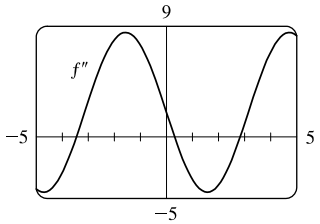
From the graphs of f' and f'', it appears that f is increasing on $(-0.23, 0)$ and $(1.62, \infty)$ and decreasing on $(-\infty, -0.23)$ and $(0, 1.62)$; f has a local maximum of $f(0) = 2$ and local minima of about $f(-0.23) = 1.96$ and $f(1.62) = -19.2$; f is CU on $(-\infty, -0.12)$ and $(1.24, \infty)$ and CD on $(-0.12, 1.24)$; and f has inflection points at about $(-0.12, 1.98)$ and $(1.24, -12.1)$.



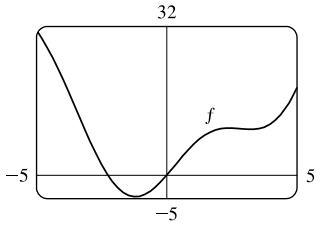


38. $f(x) = x^2 + 6.5\sin x,\ -5 \le x \le 5 \;\Rightarrow\; f'(x) = 2x + 6.5\cos x \;\Rightarrow\; f''(x) = 2 - 6.5\sin x.$ $f(x) = 0 \;\Leftrightarrow$ $x \approx -2.25$ and $x = 0$; $f'(x) = 0 \;\Leftrightarrow\; x \approx -1.19, 2.40, 3.24$; $f''(x) = 0 \;\Leftrightarrow\; x \approx -3.45, 0.31, 2.83$.

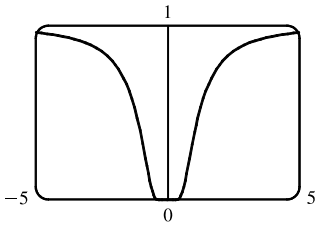




From the graphs of f' and f'', it appears that f is decreasing on $(-5, -1.19)$ and $(2.40, 3.24)$ and increasing on $(-1.19, 2.40)$ and $(3.24, 5)$; f has a local maximum of about $f(2.40) = 10.15$ and local minima of about $f(-1.19) = -4.62$ and $f(3.24) = 9.86$; f is CU on $(-3.45, 0.31)$ and $(2.83, 5)$ and CD on $(-5, -3.45)$ and $(0.31, 2.83)$; and f has inflection points at about $(-3.45, 13.93)$, $(0.31, 2.10)$, and $(2.83, 10.00)$.



39.



From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$.

$f(x) = e^{-1/x^2} \;\Rightarrow\; f'(x) = 2x^{-3}e^{-1/x^2} \;\Rightarrow$

$f''(x) = 2[x^{-3}(2x^{-3})e^{-1/x^2} + e^{-1/x^2}(-3x^{-4})] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right)$.

This is 0 when $2 - 3x^2 = 0 \;\Leftrightarrow\; x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

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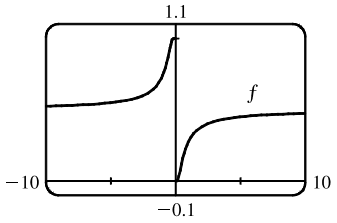
40. (a)



(b) $f(x) = \dfrac{1}{1+e^{1/x}}$.

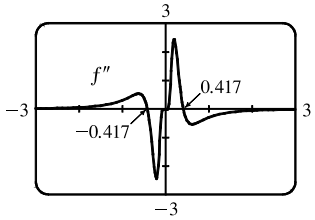
$\lim\limits_{x\to\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$, $\lim\limits_{x\to-\infty} f(x) = \dfrac{1}{1+1} = \dfrac{1}{2}$,

as $x \to 0^+$, $1/x \to \infty$, so $e^{1/x} \to \infty \quad \Rightarrow \quad \lim\limits_{x\to0^+} f(x) = 0$,

as $x \to 0^-$, $1/x \to -\infty$, so $e^{1/x} \to 0 \quad \Rightarrow \quad \lim\limits_{x\to0^-} f(x) = \dfrac{1}{1+0} = 1$

(c) From the graph of f, estimates for the IP are $(-0.4, 0.9)$ and $(0.4, 0.08)$.

(d) $f''(x) = -\dfrac{e^{1/x}[e^{1/x}(2x-1)+2x+1]}{x^4(e^{1/x}+1)^3}$



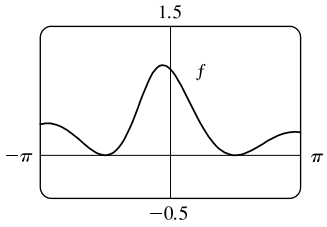
(e) From the graph, we see that f'' changes sign at $x = \pm 0.417$ ($x = 0$ is not in the domain of f). IP are approximately $(0.417, 0.083)$ and $(-0.417, 0.917)$.

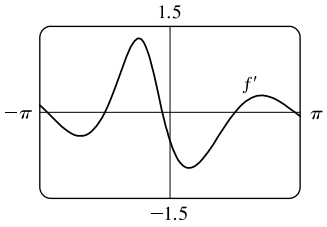
41. $f(x) = \dfrac{\cos^2 x}{\sqrt{x^2+x+1}}$, $-\pi \le x \le \pi \quad \Rightarrow \quad f'(x) = -\dfrac{\cos x\,[(2x+1)\cos x + 4(x^2+x+1)\sin x]}{2(x^2+x+1)^{3/2}} \quad \Rightarrow$

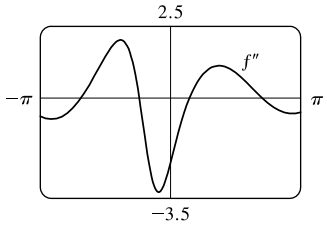
$f''(x) = -\dfrac{(8x^4+16x^3+16x^2+8x+9)\cos^2 x - 8(x^2+x+1)(2x+1)\sin x\,\cos x - 8(x^2+x+1)^2\sin^2 x}{4(x^2+x+1)^{5/2}}$

$f(x) = 0 \quad \Leftrightarrow \quad x = \pm\frac{\pi}{2}$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx -2.96, -1.57, -0.18, 1.57, 3.01$;

$f''(x) = 0 \quad \Leftrightarrow \quad x \approx -2.16, -0.75, 0.46$, and 2.21.







The x-coordinates of the maximum points are the values at which f' changes from positive to negative, that is, -2.96, -0.18, and 3.01. The x-coordinates of the minimum points are the values at which f' changes from negative to positive, that is, -1.57 and 1.57. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -2.16, -0.75, 0.46, and 2.21.

42. $f(x) = e^{-0.1x}\ln(x^2-1) \quad \Rightarrow \quad f'(x) = \dfrac{e^{-0.1x}\left[(x^2-1)\,\ln(x^2-1) - 20x\right]}{10(1-x^2)} \quad \Rightarrow$

$f''(x) = \dfrac{e^{-0.1x}\left[(x^2-1)^2\ln(x^2-1) - 40(x^3+5x^2-x+5)\right]}{100(x^2-1)^2}$.

The domain of f is $(-\infty,-1)\cup(1,\infty)$. $f(x) = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{2}$; $\quad f'(x) = 0 \quad \Leftrightarrow \quad x \approx 5.87$;

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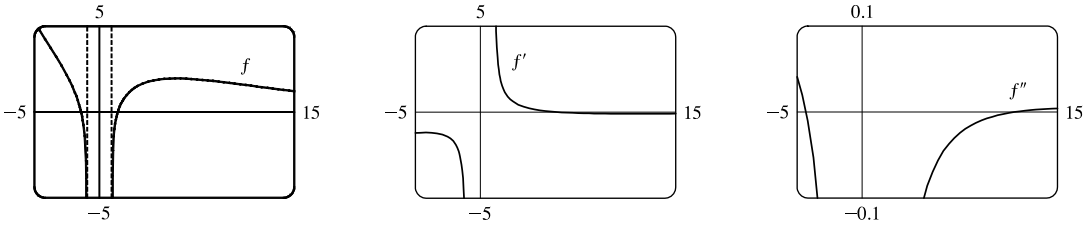
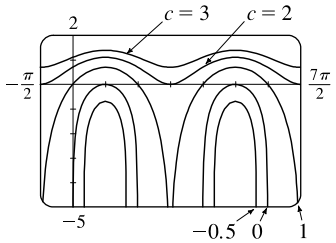
$f''(x) = 0 \quad \Leftrightarrow \quad x \approx -4.31$ and 11.74.



f' changes from positive to negative at $x \approx 5.87$, so 5.87 is the x-coordinate of the maximum point. There is no minimum point. The x-coordinates of the inflection points are the values at which f'' changes sign, that is, -4.31 and 11.74.

43. The family of functions $f(x) = \ln(\sin x + c)$ all have the same period and all have maximum values at $x = \frac{\pi}{2} + 2\pi n$. Since the domain of ln is $(0, \infty)$, f has a graph only if $\sin x + c > 0$ somewhere. Since $-1 \le \sin x \le 1$, this happens if $c > -1$, that is, f has no graph if $c \le -1$. Similarly, if $c > 1$, then $\sin x + c > 0$ and f is continuous on $(-\infty, \infty)$. As c increases, the graph of f is shifted vertically upward and flattens out. If $-1 < c \le 1$, f is defined where $\sin x + c > 0 \quad \Leftrightarrow \quad \sin x > -c \quad \Leftrightarrow \quad \sin^{-1}(-c) < x < \pi - \sin^{-1}(-c)$. Since the period is 2π, the domain of f is $\big(2n\pi + \sin^{-1}(-c), (2n+1)\pi - \sin^{-1}(-c)\big)$, n an integer.

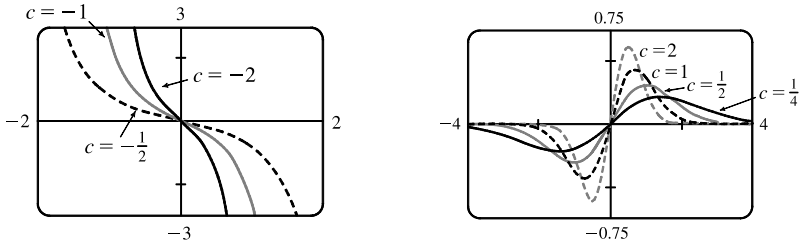


44. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate:

$$f(x) = cxe^{-cx^2} \quad \Rightarrow \quad f'(x) = c\Big[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\Big] = ce^{-cx^2}(-2cx^2 + 1)$$

This is 0 where $-2cx^2 + 1 = 0 \quad \Leftrightarrow \quad x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative solution gives a minimum and the positive solution gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f\big(\pm 1/\sqrt{2c}\big) = c\big(\pm 1/\sqrt{2c}\big)\, e^{-c\left(\pm 1/\sqrt{2c}\right)^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim_{x \to \pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim_{x \to \pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}\big(-2cx^2 + 1\big) \quad \Rightarrow$ $f''(x) = c\Big[e^{-cx^2}(-4cx) + (-2cx^2 + 1)(-2cxe^{-cx^2})\Big] = -2c^2xe^{-cx^2}(3 - 2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \quad \Leftrightarrow \quad x = \pm\sqrt{3/(2c)} \quad \Rightarrow$ IP at $\Big(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\Big)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.



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45. Let $f(x) = 3x + 2\cos x + 5$. Then $f(0) = 7 > 0$ and $f(-\pi) = -3\pi - 2 + 5 = -3\pi + 3 = -3(\pi - 1) < 0$, and since f is continuous on $\mathbb{R}$ (hence on $[-\pi, 0]$), the Intermediate Value Theorem assures us that there is at least one solution of f in $[-\pi, 0]$. Now $f'(x) = 3 - 2\sin x > 0$ implies that f is increasing on $\mathbb{R}$, so there is exactly one solution of f, and hence, exactly one real solution of the equation $3x + 2\cos x + 5 = 0$.

46. By the Mean Value Theorem, $f'(c) = \dfrac{f(4) - f(0)}{4 - 0} \quad \Leftrightarrow \quad 4f'(c) = f(4) - 1$ for some c with $0 < c < 4$. Since $2 \le f'(c) \le 5$, we have $4(2) \le 4f'(c) \le 4(5) \quad \Leftrightarrow \quad 4(2) \le f(4) - 1 \le 4(5) \quad \Leftrightarrow \quad 8 \le f(4) - 1 \le 20 \quad \Leftrightarrow$ $9 \le f(4) \le 21$.

47. Since f is continuous on $[32, 33]$ and differentiable on $(32, 33)$, then by the Mean Value Theorem there exists a number c in $(32, 33)$ such that $f'(c) = \frac{1}{5}c^{-4/5} = \dfrac{\sqrt[5]{33} - \sqrt[5]{32}}{33 - 32} = \sqrt[5]{33} - 2$, but $\frac{1}{5}c^{-4/5} > 0 \quad \Rightarrow \quad \sqrt[5]{33} - 2 > 0 \quad \Rightarrow \quad \sqrt[5]{33} > 2$. Also f' is decreasing, so that $f'(c) < f'(32) = \frac{1}{5}(32)^{-4/5} = 0.0125 \quad \Rightarrow \quad 0.0125 > f'(c) = \sqrt[5]{33} - 2 \quad \Rightarrow \quad \sqrt[5]{33} < 2.0125$. Therefore, $2 < \sqrt[5]{33} < 2.0125$.

48. Since the point $(1, 3)$ is on the curve $y = ax^3 + bx^2$, we have $3 = a(1)^3 + b(1)^2 \quad \Rightarrow \quad 3 = a + b$ **(1)**.

$y' = 3ax^2 + 2bx \quad \Rightarrow \quad y'' = 6ax + 2b$. $y'' = 0$ [for inflection points] $\quad \Leftrightarrow \quad x = \dfrac{-2b}{6a} = -\dfrac{b}{3a}$. Since we want $x = 1$, $1 = -\dfrac{b}{3a} \quad \Rightarrow \quad b = -3a$. Combining with **(1)** gives us $3 = a - 3a \quad \Leftrightarrow \quad 3 = -2a \quad \Leftrightarrow \quad a = -\frac{3}{2}$. Hence, $b = -3\left(-\frac{3}{2}\right) = \frac{9}{2}$ and the curve is $y = -\frac{3}{2}x^3 + \frac{9}{2}x^2$.

49. (a) $g(x) = f(x^2) \quad \Rightarrow \quad g'(x) = 2xf'(x^2)$ by the Chain Rule. Since $f'(x) > 0$ for all $x \neq 0$, we must have $f'(x^2) > 0$ for $x \neq 0$, so $g'(x) = 0 \quad \Leftrightarrow \quad x = 0$. Now $g'(x)$ changes sign (from negative to positive) at $x = 0$, since one of its factors, $f'(x^2)$, is positive for all x, and its other factor, $2x$, changes from negative to positive at this point, so by the First Derivative Test, f has a local and absolute minimum at $x = 0$.

(b) $g'(x) = 2xf'(x^2) \quad \Rightarrow \quad g''(x) = 2[xf''(x^2)(2x) + f'(x^2)] = 4x^2 f''(x^2) + 2f'(x^2)$ by the Product Rule and the Chain Rule. But $x^2 > 0$ for all $x \neq 0$, $f''(x^2) > 0$ [since f is CU for $x > 0$], and $f'(x^2) > 0$ for all $x \neq 0$, so since all of its factors are positive, $g''(x) > 0$ for $x \neq 0$. Whether $g''(0)$ is positive or 0 doesn't matter [since the sign of g'' does not change there]; g is concave upward on $\mathbb{R}$.

50. Call the two integers x and y. Then $x + 4y = 1000$, so $x = 1000 - 4y$. Their product is $P = xy = (1000 - 4y)y$, so our problem is to maximize the function $P(y) = 1000y - 4y^2$, where $0 < y < 250$ and y is an integer. $P'(y) = 1000 - 8y$, so $P'(y) = 0 \quad \Leftrightarrow \quad y = 125$. $P''(y) = -8 < 0$, so $P(125) = 62{,}500$ is an absolute maximum. Since the optimal y turned out to be an integer, we have found the desired pair of numbers, namely $x = 1000 - 4(125) = 500$ and $y = 125$.

51. If $B = 0$, the line is vertical and the distance from $x = -\dfrac{C}{A}$ to (x_1, y_1) is $\left|x_1 + \dfrac{C}{A}\right| = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$, so assume $B \neq 0$. The square of the distance from (x_1, y_1) to the line is $f(x) = (x - x_1)^2 + (y - y_1)^2$ where $Ax + By + C = 0$, so

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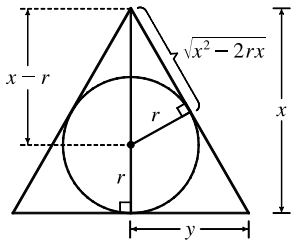
we minimize $f(x) = (x - x_1)^2 + \left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)^2 \quad \Rightarrow \quad f'(x) = 2(x - x_1) + 2\left(-\dfrac{A}{B}x - \dfrac{C}{B} - y_1\right)\left(-\dfrac{A}{B}\right)$.

$f'(x) = 0 \quad \Rightarrow \quad x = \dfrac{B^2x_1 - ABy_1 - AC}{A^2 + B^2}$ and this gives a minimum since $f''(x) = 2\left(1 + \dfrac{A^2}{B^2}\right) > 0$. Substituting this value of x into $f(x)$ and simplifying gives $f(x) = \dfrac{(Ax_1 + By_1 + C)^2}{A^2 + B^2}$, so the minimum distance is $\sqrt{f(x)} = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.

52. On the hyperbola $xy = 8$, if $d(x)$ is the distance from the point $(x, y) = (x, 8/x)$ to the point $(3, 0)$, then $[d(x)]^2 = (x - 3)^2 + 64/x^2 = f(x)$. $f'(x) = 2(x - 3) - 128/x^3 = 0 \quad \Rightarrow \quad x^4 - 3x^3 - 64 = 0 \quad \Rightarrow$ $(x - 4)\left(x^3 + x^2 + 4x + 16\right) = 0 \quad \Rightarrow \quad x = 4$ since the solution must have $x > 0$. Then $y = \frac{8}{4} = 2$, so the point is $(4, 2)$.

53.



By similar triangles, $\dfrac{y}{x} = \dfrac{r}{\sqrt{x^2 - 2rx}}$, so the area of the triangle is

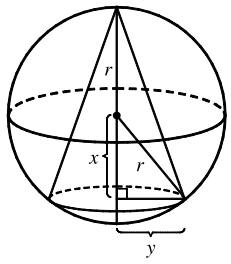
$$A(x) = \tfrac{1}{2}(2y)x = xy = \frac{rx^2}{\sqrt{x^2 - 2rx}} \quad \Rightarrow$$

$$A'(x) = \frac{2rx\sqrt{x^2 - 2rx} - rx^2(x - r)/\sqrt{x^2 - 2rx}}{x^2 - 2rx} = \frac{rx^2(x - 3r)}{(x^2 - 2rx)^{3/2}} = 0$$

when $x = 3r$.

$A'(x) < 0$ when $2r < x < 3r$, $A'(x) > 0$ when $x > 3r$. So $x = 3r$ gives a minimum and $A(3r) = \dfrac{r(9r^2)}{\sqrt{3}\,r} = 3\sqrt{3}\,r^2$.

54.



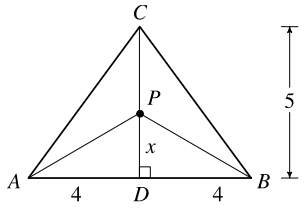
The volume of the cone is $V = \frac{1}{3}\pi y^2(r + x) = \frac{1}{3}\pi(r^2 - x^2)(r + x), \ -r \le x \le r$.

$$V'(x) = \tfrac{\pi}{3}\left[(r^2 - x^2)(1) + (r + x)(-2x)\right] = \tfrac{\pi}{3}\left[(r + x)(r - x - 2x)\right]$$
$$= \tfrac{\pi}{3}(r + x)(r - 3x) = 0 \text{ when } x = -r \text{ or } x = r/3.$$

Now $V(r) = 0 = V(-r)$, so the maximum occurs at $x = r/3$ and the volume is

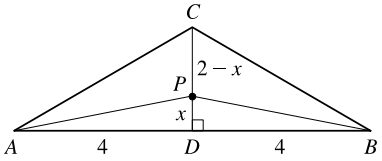
$$V\left(\frac{r}{3}\right) = \frac{\pi}{3}\left(r^2 - \frac{r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}.$$

55.



We minimize $L(x) = |PA| + |PB| + |PC| = 2\sqrt{x^2 + 16} + (5 - x)$, $0 \le x \le 5$. $L'(x) = 2x/\sqrt{x^2 + 16} - 1 = 0 \quad \Leftrightarrow \quad 2x = \sqrt{x^2 + 16} \quad \Leftrightarrow$ $4x^2 = x^2 + 16 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$. $L(0) = 13$, $L\left(\frac{4}{\sqrt{3}}\right) \approx 11.9$, $L(5) \approx 12.8$, so the minimum occurs when $x = \frac{4}{\sqrt{3}} \approx 2.3$.

56.



If $|CD| = 2$, the last part of $L(x)$ changes from $(5 - x)$ to $(2 - x)$ with $0 \le x \le 2$. But we still get $L'(x) = 0 \quad \Leftrightarrow \quad x = \frac{4}{\sqrt{3}}$, which isn't in the interval $[0, 2]$. Now $L(0) = 10$ and $L(2) = 2\sqrt{20} = 4\sqrt{5} \approx 8.9$. The minimum occurs when $P = C$.

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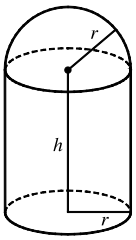
57. $v = K\sqrt{\dfrac{L}{C}+\dfrac{C}{L}} \;\Rightarrow\; \dfrac{dv}{dL} = \dfrac{K}{2\sqrt{(L/C)+(C/L)}}\left(\dfrac{1}{C}-\dfrac{C}{L^2}\right) = 0 \;\Leftrightarrow\; \dfrac{1}{C} = \dfrac{C}{L^2} \;\Leftrightarrow\; L^2 = C^2 \;\Leftrightarrow\; L = C.$

This gives the minimum velocity since $v' < 0$ for $0 < L < C$ and $v' > 0$ for $L > C$.

58.



We minimize the surface area $S = \pi r^2 + 2\pi rh + \frac{1}{2}(4\pi r^2) = 3\pi r^2 + 2\pi rh$.

Solving $V = \pi r^2 h + \frac{2}{3}\pi r^3$ for h, we get $h = \dfrac{V - \frac{2}{3}\pi r^3}{\pi r^2} = \dfrac{V}{\pi r^2} - \frac{2}{3}r$, so

$S(r) = 3\pi r^2 + 2\pi r\left[\dfrac{V}{\pi r^2} - \frac{2}{3}r\right] = \frac{5}{3}\pi r^2 + \dfrac{2V}{r}$.

$S'(r) = -\dfrac{2V}{r^2} + \frac{10}{3}\pi r = \dfrac{\frac{10}{3}\pi r^3 - 2V}{r^2} = 0 \;\Leftrightarrow\; \frac{10}{3}\pi r^3 = 2V \;\Leftrightarrow\; r^3 = \dfrac{3V}{5\pi} \;\Leftrightarrow\; r = \sqrt[3]{\dfrac{3V}{5\pi}}.$

This gives an absolute minimum since $S'(r) < 0$ for $0 < r < \sqrt[3]{\dfrac{3V}{5\pi}}$ and $S'(r) > 0$ for $r > \sqrt[3]{\dfrac{3V}{5\pi}}$. Thus,

$$h = \frac{V - \frac{2}{3}\pi \cdot \dfrac{3V}{5\pi}}{\pi\sqrt[3]{\dfrac{(3V)^2}{(5\pi)^2}}} = \frac{\left(V - \frac{2}{5}V\right)\sqrt[3]{(5\pi)^2}}{\pi\sqrt[3]{(3V)^2}} = \frac{3V\sqrt[3]{(5\pi)^2}}{5\pi\sqrt[3]{(3V)^2}} = \sqrt[3]{\frac{3V}{5\pi}} = r$$

59. Let x denote the number of \$1 decreases in ticket price. Then the ticket price is $\$12 - \$1(x)$, and the average attendance is $11{,}000 + 1000(x)$. Now the revenue per game is

$$\begin{aligned} R(x) &= (\text{price per person}) \times (\text{number of people per game}) \\ &= (12 - x)(11{,}000 + 1000x) = -1000x^2 + 1000x + 132{,}000 \end{aligned}$$

for $0 \le x \le 4$ [since the seating capacity is 15,000] $\Rightarrow$ $R'(x) = -2000x + 1000 = 0 \;\Leftrightarrow\; x = 0.5$. This is a maximum since $R''(x) = -2000 < 0$ for all x. Now we must check the value of $R(x) = (12 - x)(11{,}000 + 1000x)$ at $x = 0.5$ and at the endpoints of the domain to see which value of x gives the maximum value of R.

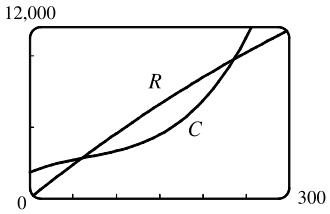
$R(0) = (12)(11{,}000) = 132{,}000$, $R(0.5) = (11.5)(11{,}500) = 132{,}250$, and $R(4) = (8)(15{,}000) = 120{,}000$. Thus, the maximum revenue of \$132,250 per game occurs when the average attendance is 11,500 and the ticket price is \$11.50.

60. (a) $C(x) = 1800 + 25x - 0.2x^2 + 0.001x^3$ and

$R(x) = xp(x) = 48.2x - 0.03x^2$.

The profit is maximized when $C'(x) = R'(x)$.

From the figure, we estimate that the tangents are parallel when $x \approx 160$.



(b) $C'(x) = 25 - 0.4x + 0.003x^2$ and $R'(x) = 48.2 - 0.06x$. $C'(x) = R'(x) \;\Rightarrow\; 0.003x^2 - 0.34x - 23.2 = 0 \;\Rightarrow$ $x_1 \approx 161.3$ $(x > 0)$. $R''(x) = -0.06$ and $C''(x) = -0.4 + 0.006x$, so $R''(x_1) = -0.06 < C''(x_1) \approx 0.57 \;\Rightarrow$ profit is maximized by producing 161 units.

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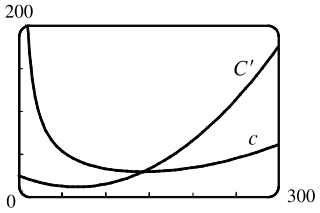
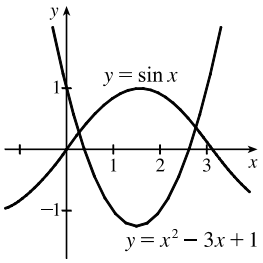
(c) $c(x) = \dfrac{C(x)}{x} = \dfrac{1800}{x} + 25 - 0.2x + 0.001x^2$ is the average cost. Since the average cost is minimized when the marginal cost equals the average cost, we graph $c(x)$ and $C'(x)$ and estimate the point of intersection. From the figure, $C'(x) = c(x) \quad \Leftrightarrow \quad x \approx 144$.



61. $f(x) = x^5 - x^4 + 3x^2 - 3x - 2 \quad \Rightarrow \quad f'(x) = 5x^4 - 4x^3 + 6x - 3$, so $x_{n+1} = x_n - \dfrac{x_n^5 - x_n^4 + 3x_n^2 - 3x_n - 2}{5x_n^4 - 4x_n^3 + 6x_n - 3}$.

Now $x_1 = 1 \quad \Rightarrow \quad x_2 = 1.5 \quad \Rightarrow \quad x_3 \approx 1.343860 \quad \Rightarrow \quad x_4 \approx 1.300320 \quad \Rightarrow \quad x_5 \approx 1.297396 \quad \Rightarrow$ $x_6 \approx 1.297383 \approx x_7$, so the solution in $[1, 2]$ is 1.297383, to six decimal places.

62. Graphing $y = \sin x$ and $y = x^2 - 3x + 1$ shows that there are two solutions, one about 0.3 and the other about 2.8.



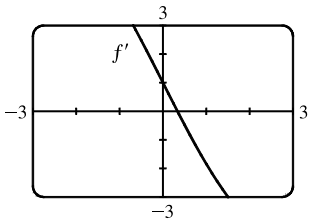
$f(x) = \sin x - x^2 + 3x - 1 \quad \Rightarrow \quad f'(x) = \cos x - 2x + 3 \quad \Rightarrow$

$x_{n+1} = x_n - \dfrac{\sin x_n - x_n^2 + 3x_n - 1}{\cos x_n - 2x_n + 3}$. Now $x_1 = 0.3 \quad \Rightarrow$

$x_2 \approx 0.268552 \quad \Rightarrow \quad x_3 \approx 0.268881 \approx x_4$ and $x_1 = 2.8 \quad \Rightarrow$

$x_2 \approx 2.770354 \quad \Rightarrow \quad x_3 \approx 2.770058 \approx x_4$, so to six decimal places, the solutions are 0.268881 and 2.770058.

63. $f(t) = \cos t + t - t^2 \quad \Rightarrow \quad f'(t) = -\sin t + 1 - 2t$. $f'(t)$ exists for all t, so to find the maximum of f, we can examine the zeros of f'. From the graph of f', we see that a good choice for t_1 is $t_1 = 0.3$. Use $g(t) = -\sin t + 1 - 2t$ and $g'(t) = -\cos t - 2$ to obtain $t_2 \approx 0.33535293$, $t_3 \approx 0.33541803 \approx t_4$. Since $f''(t) = -\cos t - 2 < 0$ for all t, $f(0.33541803) \approx 1.16718557$ is the absolute maximum.



64. $y = f(x) = x \sin x, \quad 0 \le x \le 2\pi$. **A.** $D = [0, 2\pi]$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $\sin x = 0 \quad \Leftrightarrow \quad x = 0, \pi$, or 2π. **C.** There is no symmetry on D, but if f is defined for all real numbers x, then f is an even function. **D.** No asymptote **E.** $f'(x) = x \cos x + \sin x$. To find critical numbers in $(0, 2\pi)$, we graph f' and see that there are two critical numbers, about 2 and 4.9. To find them more precisely, we use Newton's method, setting

$g(x) = f'(x) = x \cos x + \sin x$, so that $g'(x) = f''(x) = 2 \cos x - x \sin x$ and $x_{n+1} = x_n - \dfrac{x_n \cos x_n + \sin x_n}{2 \cos x_n - x_n \sin x_n}$.

$x_1 = 2 \quad \Rightarrow \quad x_2 \approx 2.029048$, $x_3 \approx 2.028758 \approx x_4$ and $x_1 = 4.9 \quad \Rightarrow \quad x_2 \approx 4.913214$, $x_3 \approx 4.913180 \approx x_4$, so the critical numbers, to six decimal places, are $r_1 = 2.028758$ and $r_2 = 4.913180$. By checking sample values of f' in $(0, r_1)$, (r_1, r_2), and $(r_2, 2\pi)$, we see that f is increasing on $(0, r_1)$, decreasing on (r_1, r_2), and increasing on $(r_2, 2\pi)$. **F.** Local maximum value $f(r_1) \approx 1.819706$, local minimum value $f(r_2) \approx -4.814470$. **G.** $f''(x) = 2 \cos x - x \sin x$. To find

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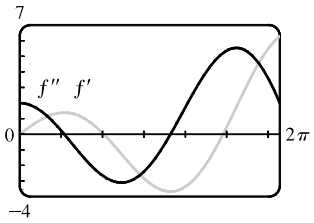
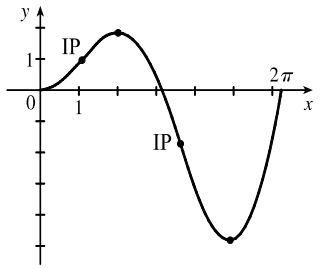
points where $f''(x) = 0$, we graph f'' and find that $f''(x) = 0$ at about 1 and 3.6. To find the values more precisely, we use Newton's method. Set $h(x) = f''(x) = 2\cos x - x\sin x$. Then $h'(x) = -3\sin x - x\cos x$, so

$x_{n+1} = x_n - \dfrac{2\cos x_n - x_n \sin x_n}{-3\sin x_n - x_n \cos x_n}$. $x_1 = 1 \Rightarrow x_2 \approx 1.078028$, $x_3 \approx 1.076874 \approx x_4$ and $x_1 = 3.6 \Rightarrow$ $x_2 \approx 3.643996$, $x_3 \approx 3.643597 \approx x_4$, so the zeros of f'', to six decimal places, are $r_3 = 1.076874$ and $r_4 = 3.643597$. By checking sample values of f'' in $(0, r_3)$, (r_3, r_4), and $(r_4, 2\pi)$, we see that f is CU on $(0, r_3)$, CD on (r_3, r_4), and CU on $(r_4, 2\pi)$. f has inflection points at $(r_3, f(r_3) \approx 0.948166)$ and $(r_4, f(r_4) \approx -1.753240)$.



H.



65. $f(x) = 4\sqrt{x} - 6x^2 + 3 = 4x^{1/2} - 6x^2 + 3 \Rightarrow F(x) = 4\left(\frac{2}{3}x^{3/2}\right) - 6\left(\frac{1}{3}x^3\right) + 3x + C = \frac{8}{3}x^{3/2} - 2x^3 + 3x + C$

66. $g(x) = \dfrac{1}{x} + \dfrac{1}{x^2+1} \Rightarrow G(x) = \begin{cases} \ln x + \tan^{-1} x + C_1 & \text{if } x > 0 \\ \ln(-x) + \tan^{-1} x + C_2 & \text{if } x < 0 \end{cases}$

67. $f(t) = 2\sin t - 3e^t \Rightarrow F(t) = -2\cos t - 3e^t + C$

68. $f(x) = x^{-3} + \cosh x \Rightarrow F(x) = \begin{cases} -1/(2x^2) + \sinh x + C_1 & \text{if } x > 0 \\ -1/(2x^2) + \sinh x + C_2 & \text{if } x < 0 \end{cases}$

69. $f'(t) = 2t - 3\sin t \Rightarrow f(t) = t^2 + 3\cos t + C$.

$f(0) = 3 + C$ and $f(0) = 5 \Rightarrow C = 2$, so $f(t) = t^2 + 3\cos t + 2$.

70. $f'(u) = \dfrac{u^2 + \sqrt{u}}{u} = u + u^{-1/2} \Rightarrow f(u) = \frac{1}{2}u^2 + 2u^{1/2} + C$.

$f(1) = \frac{1}{2} + 2 + C$ and $f(1) = 3 \Rightarrow C = \frac{1}{2}$, so $f(u) = \frac{1}{2}u^2 + 2\sqrt{u} + \frac{1}{2}$.

71. $f''(x) = 1 - 6x + 48x^2 \Rightarrow f'(x) = x - 3x^2 + 16x^3 + C$. $f'(0) = C$ and $f'(0) = 2 \Rightarrow C = 2$, so $f'(x) = x - 3x^2 + 16x^3 + 2$ and hence, $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + D$.

$f(0) = D$ and $f(0) = 1 \Rightarrow D = 1$, so $f(x) = \frac{1}{2}x^2 - x^3 + 4x^4 + 2x + 1$.

72. $f''(x) = 5x^3 + 6x^2 + 2 \Rightarrow f'(x) = \frac{5}{4}x^4 + 2x^3 + 2x + C \Rightarrow f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 + Cx + D$. Now $f(0) = D$ and $f(0) = 3$, so $D = 3$. Also, $f(1) = \frac{1}{4} + \frac{1}{2} + 1 + C + 3 = C + \frac{19}{4}$ and $f(1) = -2$, so $C + \frac{19}{4} = -2 \Rightarrow C = -\frac{27}{4}$. Thus, $f(x) = \frac{1}{4}x^5 + \frac{1}{2}x^4 + x^2 - \frac{27}{4}x + 3$.

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73. $v(t) = s'(t) = 2t - \dfrac{1}{1+t^2} \quad \Rightarrow \quad s(t) = t^2 - \tan^{-1} t + C.$

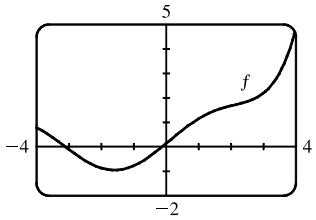
$s(0) = 0 - 0 + C = C$ and $s(0) = 1 \quad \Rightarrow \quad C = 1$, so $s(t) = t^2 - \tan^{-1} t + 1.$

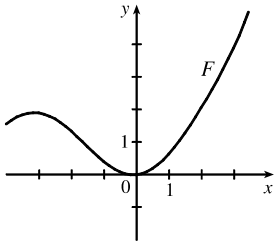
74. $a(t) = v'(t) = \sin t + 3\cos t \quad \Rightarrow \quad v(t) = -\cos t + 3\sin t + C.$

$v(0) = -1 + 0 + C$ and $v(0) = 2 \quad \Rightarrow \quad C = 3$, so $v(t) = -\cos t + 3\sin t + 3$ and $s(t) = -\sin t - 3\cos t + 3t + D.$

$s(0) = -3 + D$ and $s(0) = 0 \quad \Rightarrow \quad D = 3$, and $s(t) = -\sin t - 3\cos t + 3t + 3.$

75. (a) Since f is 0 just to the left of the y-axis, we must have a minimum of F at the same place since we are increasing through $(0, 0)$ on F. There must be a local maximum to the left of $x = -3$, since f changes from positive to negative there.





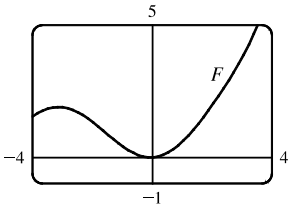
(b) $f(x) = 0.1e^x + \sin x \quad \Rightarrow$

$F(x) = 0.1e^x - \cos x + C$. $F(0) = 0 \quad \Rightarrow$

$0.1 - 1 + C = 0 \quad \Rightarrow \quad C = 0.9$, so

$F(x) = 0.1e^x - \cos x + 0.9.$

(c)



76. $f(x) = x^4 + x^3 + cx^2 \quad \Rightarrow \quad f'(x) = 4x^3 + 3x^2 + 2cx$. This is 0 when $x\left(4x^2 + 3x + 2c\right) = 0 \quad \Leftrightarrow \quad x = 0$ or $4x^2 + 3x + 2c = 0$. Using the quadratic formula, we find that the solutions of this last equation are $x = \dfrac{-3 \pm \sqrt{9 - 32c}}{8}$. Now if $9 - 32c < 0 \quad \Leftrightarrow \quad c > \frac{9}{32}$, then $(0, 0)$ is the only critical point, a minimum. If $c = \frac{9}{32}$, then there are two critical points (a minimum at $x = 0$, and a horizontal tangent with no maximum or minimum at $x = -\frac{3}{8}$) and if $c < \frac{9}{32}$, then there are three critical points except when $c = 0$, in which case the solution with the $+$ sign coincides with the critical point at $x = 0$. For $0 < c < \frac{9}{32}$, there is a minimum at $x = -\dfrac{3}{8} - \dfrac{\sqrt{9 - 32c}}{8}$, a maximum at $x = -\dfrac{3}{8} + \dfrac{\sqrt{9 - 32c}}{8}$, and a minimum at $x = 0$. For $c = 0$, there is a minimum at $x = -\frac{3}{4}$ and a horizontal tangent with no extremum at $x = 0$, and for $c < 0$, there is a maximum at $x = 0$, and there are minima at $x = -\dfrac{3}{8} \pm \dfrac{\sqrt{9 - 32c}}{8}$. Now we calculate $f''(x) = 12x^2 + 6x + 2c$. The solutions of this equation are $x = \dfrac{-6 \pm \sqrt{36 - 4 \cdot 12 \cdot 2c}}{24}$. So if $36 - 96c \le 0 \quad \Leftrightarrow \quad c \ge \frac{3}{8}$, then there is no inflection point. If $c < \frac{3}{8}$, then there are two inflection points at $x = -\dfrac{1}{4} \pm \dfrac{\sqrt{9 - 24c}}{12}$.

[continued]

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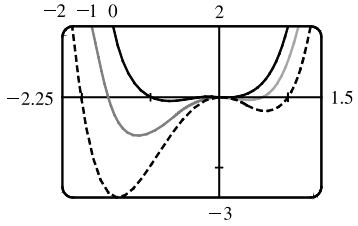
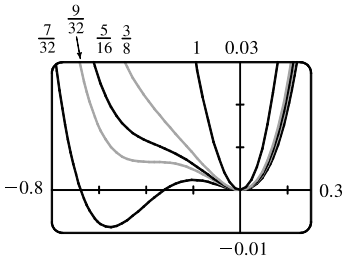
Value of c	No. of CP	No. of IP
$c < 0$	3	2
$c = 0$	2	2
$0 < c < \frac{9}{32}$	3	2
$c = \frac{9}{32}$	2	2
$\frac{9}{32} < c < \frac{3}{8}$	1	2
$c \geq \frac{3}{8}$	1	0

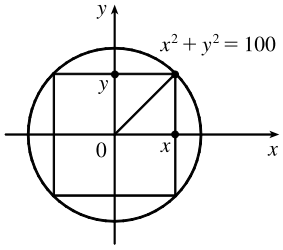




77. Choosing the positive direction to be upward, we have $a(t) = -9.8 \Rightarrow v(t) = -9.8t + v_0$, but $v(0) = 0 = v_0 \Rightarrow v(t) = -9.8t = s'(t) \Rightarrow s(t) = -4.9t^2 + s_0$, but $s(0) = s_0 = 500 \Rightarrow s(t) = -4.9t^2 + 500$. When $s = 0$, $-4.9t^2 + 500 = 0 \Rightarrow t_1 = \sqrt{\frac{500}{4.9}} \approx 10.1 \Rightarrow v(t_1) = -9.8\sqrt{\frac{500}{4.9}} \approx -98.995$ m/s. Since the canister has been designed to withstand an impact velocity of 100 m/s, the canister will *not burst*.

78. Let $s_A(t)$ and $s_B(t)$ be the position functions for cars A and B and let $f(t) = s_A(t) - s(t)$. Since A passed B twice, there must be three values of t such that $f(t) = 0$. Then by three applications of Rolle's Theorem (see Exercise 4.2.26), there is a number c such that $f''(c) = 0$. So $s_A''(c) = s_B''(c)$; that is, A and B had equal accelerations at $t = c$. We assume that f is continuous on $[0, T]$ and twice differentiable on $(0, T)$, where T is the total time of the race.

79. (a)



The cross-sectional area of the rectangular beam is

$A = 2x \cdot 2y = 4xy = 4x\sqrt{100 - x^2}$, $0 \leq x \leq 10$, so

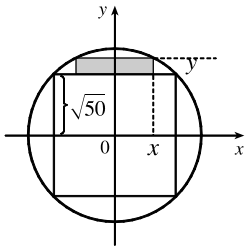
$$\frac{dA}{dx} = 4x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x) + (100 - x^2)^{1/2} \cdot 4$$

$$= \frac{-4x^2}{(100 - x^2)^{1/2}} + 4(100 - x^2)^{1/2} = \frac{4[-x^2 + (100 - x^2)]}{(100 - x^2)^{1/2}}.$$

$\frac{dA}{dx} = 0$ when $-x^2 + (100 - x^2) = 0 \Rightarrow x^2 = 50 \Rightarrow x = \sqrt{50} \approx 7.07 \Rightarrow y = \sqrt{100 - (\sqrt{50})^2} = \sqrt{50}$.

Since $A(0) = A(10) = 0$, the rectangle of maximum area is a square.

(b)



The cross-sectional area of each rectangular plank (shaded in the figure) is

$A = 2x(y - \sqrt{50}) = 2x\left[\sqrt{100 - x^2} - \sqrt{50}\right]$, $0 \leq x \leq \sqrt{50}$, so

$$\frac{dA}{dx} = 2\left(\sqrt{100 - x^2} - \sqrt{50}\right) + 2x\left(\tfrac{1}{2}\right)(100 - x^2)^{-1/2}(-2x)$$

$$= 2(100 - x^2)^{1/2} - 2\sqrt{50} - \frac{2x^2}{(100 - x^2)^{1/2}}$$

Set $\frac{dA}{dx} = 0$: $(100 - x^2) - \sqrt{50}\,(100 - x^2)^{1/2} - x^2 = 0 \Rightarrow 100 - 2x^2 = \sqrt{50}\,(100 - x^2)^{1/2} \Rightarrow$

$10{,}000 - 400x^2 + 4x^4 = 50(100 - x^2) \Rightarrow 4x^4 - 350x^2 + 5000 = 0 \Rightarrow 2x^4 - 175x^2 + 2500 = 0 \Rightarrow$

$x^2 = \frac{175 \pm \sqrt{10{,}625}}{4} \approx 69.52$ or $17.98 \Rightarrow x \approx 8.34$ or 4.24. But $8.34 > \sqrt{50}$, so $x_1 \approx 4.24 \Rightarrow$

$y - \sqrt{50} = \sqrt{100 - x_1^2} - \sqrt{50} \approx 1.99$. Each plank should have dimensions about $8\frac{1}{2}$ inches by 2 inches.

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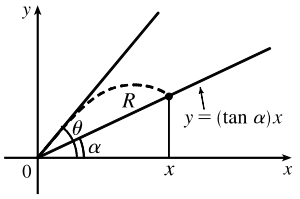
(c) From the figure in part (a), the width is $2x$ and the depth is $2y$, so the strength is $S = k(2x)(2y)^2 = 8kxy^2 = 8kx(100 - x^2) = 800kx - 8kx^3,\ 0 \le x \le 10$. $dS/dx = 800k - 24kx^2 = 0$ when $24kx^2 = 800k \ \Rightarrow\ x^2 = \frac{100}{3} \ \Rightarrow\ x = \frac{10}{\sqrt{3}} \ \Rightarrow\ y = \sqrt{\frac{200}{3}} = \frac{10\sqrt{2}}{\sqrt{3}} = \sqrt{2}\,x$. Since $S(0) = S(10) = 0$, the maximum strength occurs when $x = \frac{10}{\sqrt{3}}$. The dimensions should be $\frac{20}{\sqrt{3}} \approx 11.55$ inches by $\frac{20\sqrt{2}}{\sqrt{3}} \approx 16.33$ inches.

80. (a)



$y = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2$. The parabola intersects the line when

$(\tan\alpha)x = (\tan\theta)x - \dfrac{g}{2v^2\cos^2\theta}x^2 \ \Rightarrow$

$x = \dfrac{(\tan\theta - \tan\alpha)2v^2\cos^2\theta}{g} \ \Rightarrow$

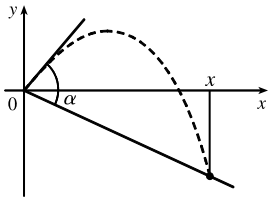
$$R(\theta) = \frac{x}{\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)\frac{2v^2\cos^2\theta}{g\cos\alpha} = \left(\frac{\sin\theta}{\cos\theta} - \frac{\sin\alpha}{\cos\alpha}\right)(\cos\theta\,\cos\alpha)\,\frac{2v^2\cos\theta}{g\cos^2\alpha}$$
$$= (\sin\theta\,\cos\alpha - \sin\alpha\,\cos\theta)\,\frac{2v^2\cos\theta}{g\cos^2\alpha} = \sin(\theta-\alpha)\,\frac{2v^2\cos\theta}{g\cos^2\alpha}$$

(b) $R'(\theta) = \dfrac{2v^2}{g\cos^2\alpha}\left[\cos\theta\cdot\cos(\theta-\alpha) + \sin(\theta-\alpha)(-\sin\theta)\right] = \dfrac{2v^2}{g\cos^2\alpha}\cos[\theta + (\theta-\alpha)]$

$= \dfrac{2v^2}{g\cos^2\alpha}\cos(2\theta-\alpha) = 0$

when $\cos(2\theta-\alpha) = 0 \ \Rightarrow\ 2\theta - \alpha = \frac{\pi}{2} \ \Rightarrow\ \theta = \dfrac{\pi/2+\alpha}{2} = \dfrac{\pi}{4} + \dfrac{\alpha}{2}$. The First Derivative Test shows that this gives a maximum value for $R(\theta)$. [This could be done without calculus by applying the formula for $\sin x\,\cos y$ to $R(\theta)$.]

(c)



Replacing α by $-\alpha$ in part (a), we get $R(\theta) = \dfrac{2v^2\cos\theta\,\sin(\theta+\alpha)}{g\cos^2\alpha}$. Proceeding as in part (b), or simply by replacing α by $-\alpha$ in the result of part (b), we see that $R(\theta)$ is maximized when $\theta = \dfrac{\pi}{4} - \dfrac{\alpha}{2}$.

81. $\lim\limits_{E\to0^+} P(E) = \lim\limits_{E\to0^+}\left(\dfrac{e^E + e^{-E}}{e^E - e^{-E}} - \dfrac{1}{E}\right)$

$= \lim\limits_{E\to0^+}\dfrac{E\left(e^E + e^{-E}\right) - 1\left(e^E - e^{-E}\right)}{\left(e^E - e^{-E}\right)E} = \lim\limits_{E\to0^+}\dfrac{Ee^E + Ee^{-E} - e^E + e^{-E}}{Ee^E - Ee^{-E}}$ [form is $\frac{0}{0}$]

$\overset{\text{H}}{=} \lim\limits_{E\to0^+}\dfrac{Ee^E + e^E\cdot1 + E\left(-e^{-E}\right) + e^{-E}\cdot1 - e^E + \left(-e^{-E}\right)}{Ee^E + e^E\cdot1 - [E(-e^{-E}) + e^{-E}\cdot1]}$

$= \lim\limits_{E\to0^+}\dfrac{Ee^E - Ee^{-E}}{Ee^E + e^E + Ee^{-E} - e^{-E}} = \lim\limits_{E\to0^+}\dfrac{e^E - e^{-E}}{e^E + \dfrac{e^E}{E} + e^{-E} - \dfrac{e^{-E}}{E}}$ [divide by E]

$= \dfrac{0}{2+L}$, where $L = \lim\limits_{E\to0^+}\dfrac{e^E - e^{-E}}{E}$ [form is $\frac{0}{0}$] $\overset{\text{H}}{=} \lim\limits_{E\to0^+}\dfrac{e^E + e^{-E}}{1} = \dfrac{1+1}{1} = 2$

Thus, $\lim\limits_{E\to0^+} P(E) = \dfrac{0}{2+2} = 0$.

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82. $\lim_{c\to0^+} s(t) = \lim_{c\to0^+}\left(\frac{m}{c}\ln\cosh\sqrt{\frac{gc}{mt}}\right) = m\lim_{c\to0^+}\frac{\ln\cosh\sqrt{ac}}{c}$ [let $a = g/(mt)$]

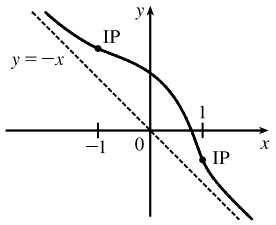
$$\overset{\text{H}}{=} m\lim_{c\to0^+}\frac{\frac{1}{\cosh\sqrt{ac}}(\sinh\sqrt{ac})\left(\frac{\sqrt{a}}{2\sqrt{c}}\right)}{1} = \frac{m\sqrt{a}}{2}\lim_{c\to0^+}\frac{\tanh\sqrt{ac}}{\sqrt{c}}$$

$$\overset{\text{H}}{=} \frac{m\sqrt{a}}{2}\lim_{c\to0^+}\frac{\operatorname{sech}^2\sqrt{ac}\left[\sqrt{a}/\left(2\sqrt{c}\right)\right]}{1/(2\sqrt{c})} = \frac{ma}{2}\lim_{c\to0^+}\operatorname{sech}^2\sqrt{ac} = \frac{ma}{2}(1)^2 = \frac{mg}{2mt} = \frac{g}{2t}$$

83. We first show that $\frac{x}{1+x^2} < \tan^{-1}x$ for $x > 0$. Let $f(x) = \tan^{-1}x - \frac{x}{1+x^2}$. Then

$f'(x) = \frac{1}{1+x^2} - \frac{1(1+x^2)-x(2x)}{(1+x^2)^2} = \frac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0$ for $x > 0$. So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1}x - \frac{x}{1+x^2}$. So $\frac{x}{1+x^2} < \tan^{-1}x$ for $0 < x$. We next show that $\tan^{-1}x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1}x$. Then $h'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1}x$. Hence, $\tan^{-1}x < x$ for $x > 0$, and we conclude that $\frac{x}{1+x^2} < \tan^{-1}x < x$ for $x > 0$.

84. If $f'(x) < 0$ for all x, $f''(x) > 0$ for $|x| > 1$, $f''(x) < 0$ for $|x| < 1$, and $\lim_{x\to\pm\infty}[f(x)+x] = 0$, then f is decreasing everywhere, concave up on $(-\infty,-1)$ and $(1,\infty)$, concave down on $(-1,1)$, and approaches the line $y = -x$ as $x \to \pm\infty$. An example of such a graph is sketched.



85. Let c be the diameter of the semicircle. Then, from the given figure and the Law of Cosines, $c^2 = a^2 + a^2 - 2a\cdot a\cdot\cos\theta$. The radius of the semicircle is $\frac{1}{2}c$, or $\frac{1}{2}\sqrt{2a^2-2a^2\cos\theta}$. The area of the figure is given by

$$\begin{aligned} A(\theta) &= \text{area of triangle} + \text{area of circle} \\ &= \tfrac{1}{2}a\cdot a\cdot\sin\theta + \tfrac{1}{2}\pi\left(\tfrac{1}{2}\sqrt{2a^2-2a^2\cos\theta}\right)^2 = \tfrac{1}{2}a^2\sin\theta + \tfrac{1}{8}\pi(2a^2-2a^2\cos\theta) \end{aligned}$$

$A'(\theta) = \frac{1}{2}a^2\cos\theta + \frac{1}{4}\pi a^2\sin\theta = 0 \;\Rightarrow\; \frac{1}{4}\pi a^2\sin\theta = -\frac{1}{2}a^2\cos\theta \;\Rightarrow\; \tan\theta = -\frac{2}{\pi} \;\Rightarrow$

$\theta = \tan^{-1}\left(-\frac{2}{\pi}\right) + n\pi$ (n an integer). We let $n = 1$ so that $0 \le \theta \le \pi$, giving $\theta = \tan^{-1}\left(-\frac{2}{\pi}\right) + \pi \approx 147.5°$.

$A''(\theta) = -\frac{1}{2}a^2\sin\theta + \frac{1}{4}\pi a^2\cos\theta < 0$ [θ. is a second-quadrant angle, so $\sin\theta > 0$ and $\cos\theta < 0$], so this value of θ gives a maximum.

86. (a) $V'(t)$ is the rate of change of the volume of the water with respect to time. $H'(t)$ is the rate of change of the height of the water with respect to time. Since the volume and the height are increasing, $V'(t)$ and $H'(t)$ are positive.

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(b) $V'(t)$ is constant, so $V''(t)$ is zero (the slope of a constant function is 0).

(c) At first, the height H of the water increases quickly because the tank is narrow. But as the sphere widens, the rate of increase of the height slows down, reaching a minimum at $t = t_2$. Thus, the height is increasing at a decreasing rate on $(0, t_2)$, so its graph is concave downward and $H''(t_1) < 0$. As the sphere narrows for $t > t_2$, the rate of increase of the height begins to increase, and the graph of H is concave upward. Therefore, $H''(t_2) = 0$ and $H''(t_3) > 0$.

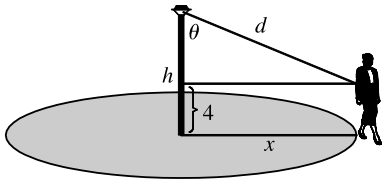
87. (a) $I = \dfrac{k\cos\theta}{d^2} = \dfrac{k(h/d)}{d^2} = k\dfrac{h}{d^3} = k\,\dfrac{h}{\left(\sqrt{40^2+h^2}\,\right)^3} = k\,\dfrac{h}{(1600+h^2)^{3/2}} \quad \Rightarrow$

$$\frac{dI}{dh} = k\,\frac{(1600+h^2)^{3/2} - h\frac{3}{2}(1600+h^2)^{1/2}\cdot 2h}{[(1600+h^2)^{3/2}]^2} = \frac{k(1600+h^2)^{1/2}(1600+h^2-3h^2)}{(1600+h^2)^3}$$

$$= \frac{k(1600-2h^2)}{(1600+h^2)^{5/2}} \qquad [k \text{ is the constant of proportionality}]$$

Set $dI/dh = 0$: $1600 - 2h^2 = 0 \quad \Rightarrow \quad h^2 = 800 \quad \Rightarrow \quad h = \sqrt{800} = 20\sqrt{2}$. By the First Derivative Test, I has a local maximum at $h = 20\sqrt{2} \approx 28$ ft.

(b)



$\dfrac{dx}{dt} = 4$ ft/s

$$I = \frac{k\cos\theta}{d^2} = \frac{k[(h-4)/d]}{d^2} = \frac{k(h-4)}{d^3}$$

$$= \frac{k(h-4)}{[(h-4)^2+x^2]^{3/2}} = k(h-4)\left[(h-4)^2+x^2\right]^{-3/2}$$

$$\frac{dI}{dt} = \frac{dI}{dx}\cdot\frac{dx}{dt} = k(h-4)\left(-\tfrac{3}{2}\right)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 2x\cdot\frac{dx}{dt}$$

$$= k(h-4)(-3x)\left[(h-4)^2+x^2\right]^{-5/2}\cdot 4 = \frac{-12xk(h-4)}{[(h-4)^2+x^2]^{5/2}}$$

$$\left.\frac{dI}{dt}\right|_{x=40} = -\frac{480k(h-4)}{[(h-4)^2+1600]^{5/2}}$$

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PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}\left(1 - 2x^2\right) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. Let $f(x) = \sin x - \cos x$ on $[0, 2\pi]$ since f has period 2π. $f'(x) = \cos x + \sin x = 0 \quad \Leftrightarrow \quad \cos x = -\sin x \quad \Leftrightarrow$ $\tan x = -1 \quad \Leftrightarrow \quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$. Evaluating f at its critical numbers and endpoints, we get $f(0) = -1$, $f\left(\frac{3\pi}{4}\right) = \sqrt{2}$, $f\left(\frac{7\pi}{4}\right) = -\sqrt{2}$, and $f(2\pi) = -1$. So f has absolute maximum value $\sqrt{2}$ and absolute minimum value $-\sqrt{2}$. Thus, $-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \quad \Rightarrow \quad |\sin x - \cos x| \leq \sqrt{2}$.

3. $f(x)$ has the form $e^{g(x)}$, so it will have an absolute maximum (minimum) where g has an absolute maximum (minimum).

$$g(x) = 10|x-2| - x^2 = \begin{cases} 10(x-2) - x^2 & \text{if } x - 2 > 0 \\ 10[-(x-2)] - x^2 & \text{if } x - 2 < 0 \end{cases} = \begin{cases} -x^2 + 10x - 20 & \text{if } x > 2 \\ -x^2 - 10x + 20 & \text{if } x < 2 \end{cases} \quad \Rightarrow$$

$$g'(x) = \begin{cases} -2x + 10 & \text{if } x > 2 \\ -2x - 10 & \text{if } x < 2 \end{cases}$$

$g'(x) = 0$ if $x = -5$ or $x = 5$, and $g'(2)$ does not exist, so the critical numbers of g are -5, 2, and 5. Since $g''(x) = -2$ for all $x \neq 2$, g is concave downward on $(-\infty, 2)$ and $(2, \infty)$, and g will attain its absolute maximum at one of the critical numbers. Since $g(-5) = 45$, $g(2) = -4$, and $g(5) = 5$, we see that $f(-5) = e^{45}$ is the absolute maximum value of f. Also, $\lim_{x \to \infty} g(x) = -\infty$, so $\lim_{x \to \infty} f(x) = \lim_{x \to \infty} e^{g(x)} = 0$. But $f(x) > 0$ for all x, so there is no absolute minimum value of f.

4. $x^2y^2\left(4 - x^2\right)\left(4 - y^2\right) = x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) = f(x)f(y)$, where $f(t) = t^2\left(4 - t^2\right)$. We will show that $0 \leq f(t) \leq 4$ for $|t| \leq 2$, which gives $0 \leq f(x)f(y) \leq 16$ for $|x| \leq 2$ and $|y| \leq 2$.

$f(t) = 4t^2 - t^4 \quad \Rightarrow \quad f'(t) = 8t - 4t^3 = 4t\left(2 - t^2\right) = 0 \quad \Rightarrow \quad t = 0$ or $\pm\sqrt{2}$.

$f(0) = 0$, $f\left(\pm\sqrt{2}\right) = 2(4 - 2) = 4$, and $f(2) = 0$. So 0 is the absolute minimum value of $f(t)$ on $[-2, 2]$ and 4 is the absolute maximum value of $f(t)$ on $[-2, 2]$. We conclude that $0 \leq f(t) \leq 4$ for $|t| \leq 2$ and hence, $0 \leq f(x)f(y) \leq 4^2$ or $0 \leq x^2\left(4 - x^2\right)y^2\left(4 - y^2\right) \leq 16$.

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5. $y=\dfrac{\sin x}{x} \;\Rightarrow\; y'=\dfrac{x\cos x-\sin x}{x^2} \;\Rightarrow\; y''=\dfrac{-x^2\sin x-2x\cos x+2\sin x}{x^3}$. If (x, y) is an inflection point, then $y''=0 \;\Rightarrow\; (2-x^2)\sin x=2x\cos x \;\Rightarrow\; (2-x^2)^2\sin^2 x=4x^2\cos^2 x \;\Rightarrow$

$(2-x^2)^2\sin^2 x=4x^2(1-\sin^2 x) \;\Rightarrow\; (4-4x^2+x^4)\sin^2 x=4x^2-4x^2\sin^2 x \;\Rightarrow$

$(4+x^4)\sin^2 x=4x^2 \;\Rightarrow\; (x^4+4)\dfrac{\sin^2 x}{x^2}=4 \;\Rightarrow\; y^2(x^4+4)=4$ since $y=\dfrac{\sin x}{x}$.

6. Let $P\left(a, 1-a^2\right)$ be the point of contact. The equation of the tangent line at P is $y-\left(1-a^2\right)=(-2a)(x-a) \;\Rightarrow$

$y-1+a^2=-2ax+2a^2 \;\Rightarrow\; y=-2ax+a^2+1$. To find the x-intercept, put $y=0$: $2ax=a^2+1 \;\Rightarrow$

$x=\dfrac{a^2+1}{2a}$. To find the y-intercept, put $x=0$: $y=a^2+1$. Therefore, the area of the triangle is

$\dfrac{1}{2}\left(\dfrac{a^2+1}{2a}\right)\left(a^2+1\right)=\dfrac{\left(a^2+1\right)^2}{4a}$. Therefore, we minimize the function $A(a)=\dfrac{\left(a^2+1\right)^2}{4a}$, $a>0$.

$A'(a)=\dfrac{(4a)2\left(a^2+1\right)(2a)-\left(a^2+1\right)^2(4)}{16a^2}=\dfrac{\left(a^2+1\right)[4a^2-\left(a^2+1\right)]}{4a^2}=\dfrac{\left(a^2+1\right)\left(3a^2-1\right)}{4a^2}$.

$A'(a)=0$ when $3a^2-1=0 \;\Rightarrow\; a=\frac{1}{\sqrt{3}}$. $A'(a)<0$ for $a<\frac{1}{\sqrt{3}}$, $A'(a)>0$ for $a>\frac{1}{\sqrt{3}}$. So by the First Derivative Test, there is an absolute minimum when $a=\frac{1}{\sqrt{3}}$. The required point is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$ and the corresponding minimum area is $A\left(\frac{1}{\sqrt{3}}\right)=\frac{4\sqrt{3}}{9}$.

7. Let $L=\lim\limits_{x\to 0}\dfrac{ax^2+\sin bx+\sin cx+\sin dx}{3x^2+5x^4+7x^6}$. Now L has the indeterminate form of type $\frac{0}{0}$, so we can apply l'Hospital's Rule. $L=\lim\limits_{x\to 0}\dfrac{2ax+b\cos bx+c\cos cx+d\cos dx}{6x+20x^3+42x^5}$. The denominator approaches 0 as $x\to 0$, so the numerator must also approach 0 (because the limit exists). But the numerator approaches $0+b+c+d$, so $b+c+d=0$. Apply l'Hospital's Rule again. $L=\lim\limits_{x\to 0}\dfrac{2a-b^2\sin bx-c^2\sin cx-d^2\sin dx}{6+60x^2+210x^4}=\dfrac{2a-0}{6+0}=\dfrac{2a}{6}$, which must equal 8.

$\dfrac{2a}{6}=8 \;\Rightarrow\; a=24$. Thus, $a+b+c+d=a+(b+c+d)=24+0=24$.

8. We first present some preliminary results that we will invoke when calculating the limit.

(1) If $y=(1+ax)^x$, then $\ln y=x\ln(1+ax)$, and $\lim\limits_{x\to 0^+}\ln y=\lim\limits_{x\to 0^+}x\ln(1+ax)=0$. Thus, $\lim\limits_{x\to 0^+}(1+ax)^x=e^0=1$.

(2) If $y=(1+ax)^x$, then $\ln y=x\ln(1+ax)$, and implicitly differentiating gives us $\dfrac{y'}{y}=x\cdot\dfrac{a}{1+ax}+\ln(1+ax) \;\Rightarrow$

$y'=y\left[\dfrac{ax}{1+ax}+\ln(1+ax)\right]$. Thus, $y=(1+ax)^x \;\Rightarrow\; y'=(1+ax)^x\left[\dfrac{ax}{1+ax}+\ln(1+ax)\right]$.

(3) If $y=\dfrac{ax}{1+ax}$, then $y'=\dfrac{(1+ax)a-ax(a)}{(1+ax)^2}=\dfrac{a+a^2x-a^2x}{(1+ax)^2}=\dfrac{a}{(1+ax)^2}$.

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$$\lim_{x\to\infty}\frac{(x+2)^{1/x}-x^{1/x}}{(x+3)^{1/x}-x^{1/x}} = \lim_{x\to\infty}\frac{x^{1/x}[(1+2/x)^{1/x}-1]}{x^{1/x}[(1+3/x)^{1/x}-1]} \qquad \text{[factor out } x^{1/x}\text{]}$$

$$= \lim_{x\to\infty}\frac{(1+2/x)^{1/x}-1}{(1+3/x)^{1/x}-1}$$

$$= \lim_{t\to0^+}\frac{(1+2t)^{t}-1}{(1+3t)^{t}-1} \qquad \text{[let } t=1/x\text{, form } 0/0 \text{ by } \textbf{(1)}\text{]}$$

$$\overset{\text{H}}{=} \lim_{t\to0^+}\frac{(1+2t)^{t}\left[\dfrac{2t}{1+2t}+\ln(1+2t)\right]}{(1+3t)^{t}\left[\dfrac{3t}{1+3t}+\ln(1+3t)\right]} \qquad \text{[by } \textbf{(2)}\text{]}$$

$$= \lim_{t\to0^+}\frac{(1+2t)^{t}}{(1+3t)^{t}}\cdot\lim_{t\to0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)}$$

$$= \frac{1}{1}\cdot\lim_{t\to0^+}\frac{\dfrac{2t}{1+2t}+\ln(1+2t)}{\dfrac{3t}{1+3t}+\ln(1+3t)} \qquad \text{[by } \textbf{(1)}\text{, now form } 0/0\text{]}$$

$$\overset{\text{H}}{=} \lim_{t\to0^+}\frac{\dfrac{2}{(1+2t)^2}+\dfrac{2}{1+2t}}{\dfrac{3}{(1+3t)^2}+\dfrac{3}{1+3t}} \qquad \text{[by } \textbf{(3)}\text{]}$$

$$= \frac{2+2}{3+3} = \frac{4}{6} = \frac{2}{3}$$

9. Differentiating $x^2+xy+y^2=12$ implicitly with respect to x gives $2x+y+x\,\dfrac{dy}{dx}+2y\,\dfrac{dy}{dx}=0$, so $\dfrac{dy}{dx}=-\dfrac{2x+y}{x+2y}$.

At a highest or lowest point, $\dfrac{dy}{dx}=0 \quad\Leftrightarrow\quad y=-2x$. Substituting $-2x$ for y in the original equation gives $x^2+x(-2x)+(-2x)^2=12$, so $3x^2=12$ and $x=\pm2$. If $x=2$, then $y=-2x=-4$, and if $x=-2$ then $y=4$. Thus, the highest and lowest points are $(-2,4)$ and $(2,-4)$.

10. Since

$$\frac{f(x+n)-f(x)}{n}=f'(x) \qquad \textbf{(1)}$$

holds for all real numbers x and all positive integers n, we have

$$\frac{f(x+n)-f(x)}{n}=\frac{f(x+2n)-f(x)}{2n}$$

for every real number x. It follows that

$$f(x+2n)-2f(x+n)=-f(x) \qquad \textbf{(2)}$$

Now, again from **(1)**, we can write

$$nf'(x)=f(x+n)-f(x)$$

The right-hand side of this equation is differentiable by hypothesis, so the left-hand side is also differentiable. Differentiating

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and then using **(1)** again — twice this time — we get

$$\begin{aligned} nf''(x) &= f'(x+n) - f'(x) \\ &= \frac{f(x+2n) - f(x+n)}{n} - \frac{f(x+n) - f(x)}{n} = \frac{f(x+2n) - 2f(x+n) + f(x)}{n} \end{aligned}$$

Rearranging this last equation and simplifying using **(2)**, we get

$$n^2 f''(x) = f(x+2n) - 2f(x+n) + f(x) = -f(x) + f(x) = 0$$

Thus, $f''(x) = 0$ for all x, so f is a linear function.

11. (a) $y = x^2 \;\Rightarrow\; y' = 2x$, so the slope of the tangent line at $P(a, a^2)$ is $2a$ and the slope of the normal line is $-\dfrac{1}{2a}$ for $a \neq 0$. An equation of the normal line is $y - a^2 = -\dfrac{1}{2a}(x - a)$. Substitute x^2 for y to find the x-coordinates of the two points of intersection of the parabola and the normal line. $x^2 - a^2 = -\dfrac{x}{2a} + \dfrac{1}{2} \;\Leftrightarrow\; x^2 + \left(\dfrac{1}{2a}\right)x - \dfrac{1}{2} - a^2 = 0$. We know that a is a root of this quadratic equation, so $x - a$ is a factor, and we have $(x - a)\left(x + \dfrac{1}{2a} + a\right) = 0$, and hence, $x = -a - \dfrac{1}{2a}$ is the x-coordinate of the point Q. We want to minimize the y-coordinate of Q, which is $\left(-a - \dfrac{1}{2a}\right)^2 = a^2 + 1 + \dfrac{1}{4a^2} = y(a)$. Now $y'(a) = 2a - \dfrac{1}{2a^3} = \dfrac{4a^4 - 1}{2a^3} = \dfrac{(2a^2+1)(2a^2-1)}{2a^3} = 0 \;\Rightarrow$ $a = \dfrac{1}{\sqrt{2}}$ for $a > 0$. Since $y''(a) = 2 + \dfrac{3}{2a^4} > 0$, we see that $a = \dfrac{1}{\sqrt{2}}$ gives us the minimum value of the y-coordinate of Q.

(b) The square S of the distance from $P(a, a^2)$ to $Q\left(-a - \dfrac{1}{2a}, \left(-a - \dfrac{1}{2a}\right)^2\right)$ is given by

$$\begin{aligned} S &= \left(-a - \frac{1}{2a} - a\right)^2 + \left[\left(-a - \frac{1}{2a}\right)^2 - a^2\right]^2 = \left(-2a - \frac{1}{2a}\right)^2 + \left[\left(a^2 + 1 + \frac{1}{4a^2}\right) - a^2\right]^2 \\ &= \left(4a^2 + 2 + \frac{1}{4a^2}\right) + \left(1 + \frac{1}{4a^2}\right)^2 = \left(4a^2 + 2 + \frac{1}{4a^2}\right) + 1 + \frac{2}{4a^2} + \frac{1}{16a^4} \\ &= 4a^2 + 3 + \frac{3}{4a^2} + \frac{1}{16a^4} \end{aligned}$$

$S' = 8a - \dfrac{6}{4a^3} - \dfrac{4}{16a^5} = 8a - \dfrac{3}{2a^3} - \dfrac{1}{4a^5} = \dfrac{32a^6 - 6a^2 - 1}{4a^5} = \dfrac{(2a^2-1)(4a^2+1)^2}{4a^5}$. The only real positive zero of the equation $S' = 0$ is $a = \dfrac{1}{\sqrt{2}}$. Since $S'' = 8 + \dfrac{9}{2a^4} + \dfrac{5}{4a^6} > 0$, $a = \dfrac{1}{\sqrt{2}}$ corresponds to the shortest possible length of the line segment PQ.

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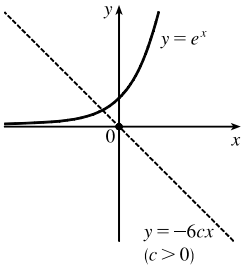
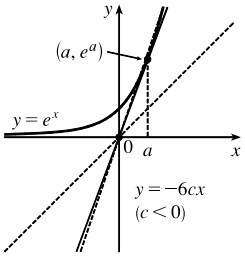
12. 

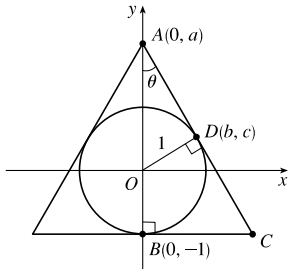
$y = cx^3 + e^x \quad \Rightarrow \quad y' = 3cx^2 + e^x \quad \Rightarrow \quad y'' = 6cx + e^x$. The curve will have inflection points when y'' changes sign. $y'' = 0 \quad \Rightarrow \quad -6cx = e^x$, so y'' will change sign when the line $y = -6cx$ intersects the curve $y = e^x$ (but is not tangent to it). Note that if $c = 0$, the curve is just $y = e^x$, which has no inflection point.

The first figure shows that for $c > 0$, $y = -6cx$ will intersect $y = e^x$ once, so $y = cx^3 + e^x$ will have one inflection point.



The second figure shows that for $c < 0$, the line $y = -6cx$ can intersect the curve $y = e^x$ in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). The tangent line at (a, e^a) has slope e^a, but from the diagram we see that the slope is $\dfrac{e^a}{a}$. So $\dfrac{e^a}{a} = e^a \quad \Rightarrow \quad a = 1$. Thus, the slope is e. The line $y = -6cx$ must have slope greater than e, so $-6c > e \quad \Rightarrow \quad c < -e/6$.

Therefore, the curve $y = cx^3 + e^x$ will have one inflection point if $c > 0$ and two inflection points if $c < -e/6$.

13. 

$\overline{AC}$ is tangent to the unit circle at D. To find the slope of $\overline{AC}$ at D, use implicit differentiation. $x^2 + y^2 = 1 \quad \Rightarrow \quad 2x + 2y\,y' = 0 \quad \Rightarrow \quad y\,y' = -x \quad \Rightarrow \quad y' = -\dfrac{x}{y}$. Thus, the tangent line at $D(b, c)$ has equation $y = -\dfrac{b}{c}\,x + a$. At D, $x = b$ and $y = c$, so $c = -\dfrac{b}{c}(b) + a \quad \Rightarrow \quad a = c + \dfrac{b^2}{c} = \dfrac{c^2 + b^2}{c} = \dfrac{1}{c}$, and hence $c = \dfrac{1}{a}$.

Since $b^2 + c^2 = 1$, $b = \sqrt{1 - c^2} = \sqrt{1 - 1/a^2} = \sqrt{\dfrac{a^2 - 1}{a^2}} = \dfrac{\sqrt{a^2 - 1}}{a}$, and now we have both b and c in terms of a. At C, $y = -1$, so $-1 = -\dfrac{b}{c}\,x + a \quad \Rightarrow \quad \dfrac{b}{c}\,x = a + 1 \quad \Rightarrow$

$x = \dfrac{c}{b}(a + 1) = \dfrac{1/a}{\sqrt{a^2 - 1}/a}(a + 1) = \dfrac{a + 1}{\sqrt{(a + 1)(a - 1)}} = \sqrt{\dfrac{a + 1}{a - 1}}$, and C has coordinates $\left(\sqrt{\dfrac{a + 1}{a - 1}}, -1\right)$. Let S be the square of the distance from A to C. Then $S(a) = \left(0 - \sqrt{\dfrac{a + 1}{a - 1}}\right)^2 + (a + 1)^2 = \dfrac{a + 1}{a - 1} + (a + 1)^2 \quad \Rightarrow$

$$S'(a) = \frac{(a - 1)(1) - (a + 1)(1)}{(a - 1)^2} + 2(a + 1) = \frac{-2 + 2(a + 1)(a - 1)^2}{(a - 1)^2}$$

$$= \frac{-2 + 2(a^3 - a^2 - a + 1)}{(a - 1)^2} = \frac{2a^3 - 2a^2 - 2a}{(a - 1)^2} = \frac{2a(a^2 - a - 1)}{(a - 1)^2}$$

Using the quadratic formula, we find that the solutions of $a^2 - a - 1 = 0$ are $a = \dfrac{1 \pm \sqrt{5}}{2}$, so $a_1 = \dfrac{1 + \sqrt{5}}{2}$ (the "golden mean") since $a > 0$. For $1 < a < a_1$, $S'(a) < 0$, and for $a > a_1$, $S'(a) > 0$, so a_1 minimizes S.

Note: The minimum length of the equal sides is $\sqrt{S(a_1)} = \cdots = \sqrt{\dfrac{11 + 5\sqrt{5}}{2}} \approx 3.33$ and the corresponding length of the

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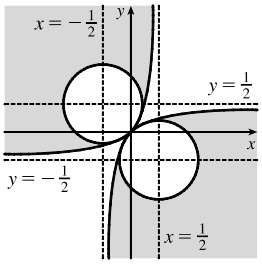
third side is $2\sqrt{\dfrac{a_1+1}{a_1-1}} = \cdots = 2\sqrt{2+\sqrt{5}} \approx 4.12$, so the triangle is *not* equilateral.

Another method: In ΔABC, $\cos\theta = \dfrac{a+1}{\overline{AC}}$, so $\overline{AC} = \dfrac{a+1}{\cos\theta}$. In ΔADO, $\sin\theta = \dfrac{1}{a}$, so $\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-1/a^2} = \dfrac{1}{a}\sqrt{a^2-1}$. Thus $\overline{AC} = \dfrac{a+1}{(1/a)\sqrt{a^2-1}} = \dfrac{a(a+1)}{\sqrt{a^2-1}} = f(a)$. Now find the minimum of f.

14. To sketch the region $\{(x,y) \mid 2xy \le |x-y| \le x^2+y^2\}$, we consider two cases.

Case 1: $x \ge y$ This is the case in which (x,y) lies on or below the line $y=x$. The double inequality becomes $2xy \le x-y \le x^2+y^2$. The right-hand inequality holds if and only if $x^2-x+y^2+y \ge 0 \quad \Leftrightarrow$ $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2 \ge \frac{1}{2} \quad \Leftrightarrow \quad (x,y)$ lies on or outside the circle with radius $\frac{1}{\sqrt{2}}$ centered at $\left(\frac{1}{2},-\frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy-x+y \le 0 \quad \Leftrightarrow \quad xy-\frac{1}{2}x+\frac{1}{2}y \le 0 \quad \Leftrightarrow$ $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) \le -\frac{1}{4} \quad \Leftrightarrow \quad (x,y)$ lies on or below the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y=\frac{1}{2}$ and $x=-\frac{1}{2}$ asymptotically.

Case 2: $y \ge x$ This is the case in which (x,y) lies on or above the line $y=x$. The double inequality becomes $2xy \le y-x \le x^2+y^2$. The right-hand inequality holds if and only if $x^2+x+y^2-y \ge 0 \quad \Leftrightarrow$ $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 \ge \frac{1}{2} \quad \Leftrightarrow \quad (x,y)$ lies on or outside the circle of radius $\frac{1}{\sqrt{2}}$ centered at $\left(-\frac{1}{2},\frac{1}{2}\right)$. The left-hand inequality holds if and only if $2xy+x-y \le 0 \quad \Leftrightarrow \quad xy+\frac{1}{2}x-\frac{1}{2}y \le 0 \quad \Leftrightarrow \quad \left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) \le -\frac{1}{4} \quad \Leftrightarrow \quad (x,y)$ lies on or above the left-hand branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$, which passes through the origin and approaches the lines $y=-\frac{1}{2}$ and $x=\frac{1}{2}$ asymptotically. Therefore, the region of interest consists of the points on or above the left branch of the hyperbola $\left(x-\frac{1}{2}\right)\left(y+\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x+\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 = \frac{1}{2}$, together with the points on or below the right branch of the hyperbola $\left(x+\frac{1}{2}\right)\left(y-\frac{1}{2}\right) = -\frac{1}{4}$ that are on or outside the circle $\left(x-\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2 = \frac{1}{2}$. Note that the inequalities are unchanged when x and y are interchanged, so the region is symmetric about the line $y=x$. So we need only have analyzed case 1 and then reflected that region about the line $y=x$, instead of considering case 2.



15. $A = \left(x_1, x_1^2\right)$ and $B = \left(x_2, x_2^2\right)$, where x_1 and x_2 are the solutions of the quadratic equation $x^2 = mx+b$. Let $P = \left(x, x^2\right)$ and set $A_1 = (x_1, 0)$, $B_1 = (x_2, 0)$, and $P_1 = (x, 0)$. Let $f(x)$ denote the area of triangle PAB. Then $f(x)$ can be expressed in terms of the areas of three trapezoids as follows:

$$\begin{aligned} f(x) &= \text{area}\,(A_1ABB_1) - \text{area}\,(A_1APP_1) - \text{area}\,(B_1BPP_1) \\ &= \tfrac{1}{2}\left(x_1^2+x_2^2\right)(x_2-x_1) - \tfrac{1}{2}\left(x_1^2+x^2\right)(x-x_1) - \tfrac{1}{2}\left(x^2+x_2^2\right)(x_2-x) \end{aligned}$$

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After expanding and canceling terms, we get

$f(x) = \frac{1}{2}\big(x_2x_1^2 - x_1x_2^2 - xx_1^2 + x_1x^2 - x_2x^2 + xx_2^2\big) = \frac{1}{2}\big[x_1^2(x_2 - x) + x_2^2(x - x_1) + x^2(x_1 - x_2)\big]$

$f'(x) = \frac{1}{2}\big[-x_1^2 + x_2^2 + 2x(x_1 - x_2)\big]. \quad f''(x) = \frac{1}{2}[2(x_1 - x_2)] = x_1 - x_2 < 0$ since $x_2 > x_1$.

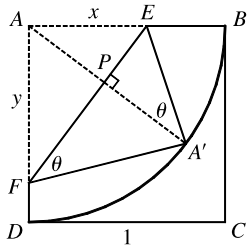
$f'(x) = 0 \quad \Rightarrow \quad 2x(x_1 - x_2) = x_1^2 - x_2^2 \quad \Rightarrow \quad x_P = \frac{1}{2}(x_1 + x_2).$

$$\begin{aligned} f(x_P) &= \tfrac{1}{2}\big(x_1^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + x_2^2\left[\tfrac{1}{2}(x_2 - x_1)\right] + \tfrac{1}{4}(x_1 + x_2)^2(x_1 - x_2)\big) \\ &= \tfrac{1}{2}\big[\tfrac{1}{2}(x_2 - x_1)\big(x_1^2 + x_2^2\big) - \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)^2\big] = \tfrac{1}{8}(x_2 - x_1)\big[2\big(x_1^2 + x_2^2\big) - \big(x_1^2 + 2x_1x_2 + x_2^2\big)\big] \\ &= \tfrac{1}{8}(x_2 - x_1)\big(x_1^2 - 2x_1x_2 + x_2^2\big) = \tfrac{1}{8}(x_2 - x_1)(x_1 - x_2)^2 = \tfrac{1}{8}(x_2 - x_1)(x_2 - x_1)^2 = \tfrac{1}{8}(x_2 - x_1)^3 \end{aligned}$$

To put this in terms of m and b, we solve the system $y = x_1^2$ and $y = mx_1 + b$, giving us $x_1^2 - mx_1 - b = 0 \quad \Rightarrow$ $x_1 = \frac{1}{2}\big(m - \sqrt{m^2 + 4b}\big)$. Similarly, $x_2 = \frac{1}{2}\big(m + \sqrt{m^2 + 4b}\big)$. The area is then $\frac{1}{8}(x_2 - x_1)^3 = \frac{1}{8}\big(\sqrt{m^2 + 4b}\big)^3$, and is attained at the point $P\big(x_P, x_P^2\big) = P\big(\frac{1}{2}m, \frac{1}{4}m^2\big)$.

Note: Another way to get an expression for $f(x)$ is to use the formula for an area of a triangle in terms of the coordinates of the vertices: $f(x) = \frac{1}{2}\big[\big(x_2x_1^2 - x_1x_2^2\big) + \big(x_1x^2 - xx_1^2\big) + \big(xx_2^2 - x_2x^2\big)\big]$.

16. Let $x = |AE|$, $y = |AF|$ as shown. The area $\mathcal{A}$ of the ΔAEF is $\mathcal{A} = \frac{1}{2}xy$. We need to find a relationship between x and y, so that we can take the derivative $d\mathcal{A}/dx$ and then find the maximum and minimum areas. Now let A' be the point on which A ends up after the fold has been performed, and let P be the intersection of AA' and EF. Note that AA' is perpendicular to EF since we are reflecting A through the line EF to get to A', and that $|AP| = |PA'|$ for the same reason.



But $|AA'| = 1$, since AA' is a radius of the circle. Since $|AP| + |PA'| = |AA'|$, we have $|AP| = \frac{1}{2}$. Another way to express the area of the triangle is $\mathcal{A} = \frac{1}{2}\,|EF|\,|AP| = \frac{1}{2}\sqrt{x^2 + y^2}\big(\frac{1}{2}\big) = \frac{1}{4}\sqrt{x^2 + y^2}$. Equating the two expressions for $\mathcal{A}$, we get

$\frac{1}{2}xy = \frac{1}{4}\sqrt{x^2 + y^2} \quad \Rightarrow \quad 4x^2y^2 = x^2 + y^2 \quad \Rightarrow \quad y^2\big(4x^2 - 1\big) = x^2 \quad \Rightarrow \quad y = x/\sqrt{4x^2 - 1}.$

(Note that we could also have derived this result from the similarity of $\triangle A'PE$ and $\triangle A'FE$; that is,

$\dfrac{|A'P|}{|PE|} = \dfrac{|A'F|}{|A'E|} \quad \Rightarrow \quad \dfrac{\frac{1}{2}}{\sqrt{x^2 - \left(\frac{1}{2}\right)^2}} = \dfrac{y}{x} \quad \Rightarrow \quad y = \dfrac{\frac{1}{2}x}{\sqrt{4x^2 - 1}\,/2} = \dfrac{x}{\sqrt{4x^2 - 1}}$.) Now we can substitute for y and

calculate $\dfrac{d\mathcal{A}}{dx}$: $\mathcal{A} = \dfrac{1}{2}\dfrac{x^2}{\sqrt{4x^2 - 1}} \quad \Rightarrow \quad \dfrac{d\mathcal{A}}{dx} = \dfrac{1}{2}\left[\dfrac{\sqrt{4x^2 - 1}\,(2x) - x^2\big(\frac{1}{2}\big)\big(4x^2 - 1\big)^{-1/2}(8x)}{4x^2 - 1}\right]$. This is 0 when

$2x\sqrt{4x^2 - 1} - 4x^3\big(4x^2 - 1\big)^{-1/2} = 0 \quad \Leftrightarrow \quad 2x\big(4x^2 - 1\big)^{-1/2}\big[\big(4x^2 - 1\big) - 2x^2\big] = 0 \quad \Rightarrow \quad \big(4x^2 - 1\big) - 2x^2 = 0$

$(x > 0) \quad \Leftrightarrow \quad 2x^2 = 1 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. So this is one possible value for an extremum. We must also test the endpoints of the interval over which x ranges. The largest value that x can attain is 1, and the smallest value of x occurs when $y = 1 \quad \Leftrightarrow$ $1 = x/\sqrt{4x^2 - 1} \quad \Leftrightarrow \quad x^2 = 4x^2 - 1 \quad \Leftrightarrow \quad 3x^2 = 1 \quad \Leftrightarrow \quad x = \frac{1}{\sqrt{3}}$. This will give the same value of $\mathcal{A}$ as will $x = 1$, since the geometric situation is the same (reflected through the line $y = x$). We calculate

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$\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \dfrac{1}{2}\dfrac{\left(1/\sqrt{2}\right)^2}{\sqrt{4\left(1/\sqrt{2}\right)^2 - 1}} = \dfrac{1}{4} = 0.25$, and $\mathcal{A}(1) = \dfrac{1}{2}\dfrac{1^2}{\sqrt{4(1)^2 - 1}} = \dfrac{1}{2\sqrt{3}} \approx 0.29$. So the maximum area is $\mathcal{A}(1) = \mathcal{A}\left(\frac{1}{\sqrt{3}}\right) = \frac{1}{2\sqrt{3}}$ and the minimum area is $\mathcal{A}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{4}$.

Another method: Use the angle θ (see diagram above) as a variable:

$\mathcal{A} = \frac{1}{2}xy = \frac{1}{2}\left(\frac{1}{2}\sec\theta\right)\left(\frac{1}{2}\csc\theta\right) = \dfrac{1}{8\sin\theta\,\cos\theta} = \dfrac{1}{4\sin 2\theta}$. $\mathcal{A}$ is minimized when $\sin 2\theta$ is maximal, that is, when $\sin 2\theta = 1 \;\Rightarrow\; 2\theta = \frac{\pi}{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$. Also note that $A'E = x = \frac{1}{2}\sec\theta \le 1 \;\Rightarrow\; \sec\theta \le 2 \;\Rightarrow\; \cos\theta \ge \frac{1}{2} \;\Rightarrow\; \theta \le \frac{\pi}{3}$, and similarly, $A'F = y = \frac{1}{2}\csc\theta \le 1 \;\Rightarrow\; \csc\theta \le 2 \;\Rightarrow\; \sin\theta \le \frac{1}{2} \;\Rightarrow\; \theta \ge \frac{\pi}{6}$.

As above, we find that $\mathcal{A}$ is maximized at these endpoints: $\mathcal{A}\left(\frac{\pi}{6}\right) = \dfrac{1}{4\sin\frac{\pi}{3}} = \dfrac{1}{2\sqrt{3}} = \dfrac{1}{4\sin\frac{2\pi}{3}} = \mathcal{A}\left(\frac{\pi}{3}\right)$;

and minimized at $\theta = \frac{\pi}{4}$: $\mathcal{A}\left(\frac{\pi}{4}\right) = \dfrac{1}{4\sin\frac{\pi}{2}} = \dfrac{1}{4}$.

17. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, $x > 0$, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$. $g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x}\cdot\dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \le e^{1/e}$. Conversely, suppose that $0 < a \le e^{1/e}$. Then $a^e \le e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.

18. If $L = \lim\limits_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

$$\ln L = \lim_{x\to\infty}\ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty}\frac{\ln(x+a) - \ln(x-a)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+a} - \dfrac{1}{x-a}}{-1/x^2}$$

$$= \lim_{x\to\infty}\left[\frac{(x-a)-(x+a)}{(x+a)(x-a)}\cdot\frac{-x^2}{1}\right] = \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = \lim_{x\to\infty}\frac{2a}{1 - a^2/x^2} = 2a$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \;\Rightarrow\; 2a = 1 \;\Rightarrow\; a = \frac{1}{2}$.

19. Note that $f(0) = 0$, so for $x \ne 0$, $\left|\dfrac{f(x) - f(0)}{x - 0}\right| = \left|\dfrac{f(x)}{x}\right| = \dfrac{|f(x)|}{|x|} \le \dfrac{|\sin x|}{|x|} = \dfrac{\sin x}{x}$.

Therefore, $|f'(0)| = \left|\lim\limits_{x\to 0}\dfrac{f(x) - f(0)}{x - 0}\right| = \lim\limits_{x\to 0}\left|\dfrac{f(x) - f(0)}{x - 0}\right| \le \lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

But $f(x) = a_1\sin x + a_2\sin 2x + \cdots + a_n\sin nx \;\Rightarrow\; f'(x) = a_1\cos x + 2a_2\cos 2x + \cdots + na_n\cos nx$, so $|f'(0)| = |a_1 + 2a_2 + \cdots + na_n| \le 1$.

Another solution: We are given that $\left|\sum_{k=1}^n a_k\sin kx\right| \le |\sin x|$. So for x close to 0, and $x \ne 0$, we have

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$\left|\sum_{k=1}^{n} a_k \frac{\sin kx}{\sin x}\right| \le 1 \quad \Rightarrow \quad \lim_{x\to 0}\left|\sum_{k=1}^{n} a_k \frac{\sin kx}{\sin x}\right| \le 1 \quad \Rightarrow \quad \left|\sum_{k=1}^{n} a_k \lim_{x\to 0}\frac{\sin kx}{\sin x}\right| \le 1$. But by l'Hospital's Rule,

$\lim_{x\to 0}\frac{\sin kx}{\sin x} = \lim_{x\to 0}\frac{k\cos kx}{\cos x} = k$, so $\left|\sum_{k=1}^{n} k a_k\right| \le 1$.

20. Let the circle have radius r, so $|OP| = |OQ| = r$, where O is the center of the circle. Now $\angle POR$ has measure $\frac{1}{2}\theta$, and $\angle OPR$ is a right angle, so $\tan\frac{1}{2}\theta = \dfrac{|PR|}{r}$ and the area of $\triangle OPR$ is $\frac{1}{2}\,|OP|\,|PR| = \frac{1}{2}r^2\tan\frac{1}{2}\theta$. The area of the sector cut by OP and OR is $\frac{1}{2}r^2\left(\frac{1}{2}\theta\right) = \frac{1}{4}r^2\theta$. Let S be the intersection of PQ and OR. Then $\sin\frac{1}{2}\theta = \dfrac{|PS|}{r}$ and $\cos\frac{1}{2}\theta = \dfrac{|OS|}{r}$, and the area of $\triangle OSP$ is $\frac{1}{2}\,|OS|\,|PS| = \frac{1}{2}\left(r\cos\frac{1}{2}\theta\right)\left(r\sin\frac{1}{2}\theta\right) = \frac{1}{2}r^2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta = \frac{1}{4}r^2\sin\theta$.

So $B(\theta) = 2\left(\frac{1}{2}r^2\tan\frac{1}{2}\theta - \frac{1}{4}r^2\theta\right) = r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)$ and $A(\theta) = 2\left(\frac{1}{4}r^2\theta - \frac{1}{4}r^2\sin\theta\right) = \frac{1}{2}r^2(\theta - \sin\theta)$. Thus,

$$\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{r^2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)} = \lim_{\theta\to 0^+}\frac{\theta-\sin\theta}{2\left(\tan\frac{1}{2}\theta - \frac{1}{2}\theta\right)} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{2\left(\frac{1}{2}\sec^2\frac{1}{2}\theta - \frac{1}{2}\right)}$$

$$= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\sec^2\frac{1}{2}\theta - 1} = \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\tan^2\frac{1}{2}\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{2\left(\tan\frac{1}{2}\theta\right)\left(\sec^2\frac{1}{2}\theta\right)\frac{1}{2}}$$

$$= \lim_{\theta\to 0^+}\frac{\sin\theta\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = \lim_{\theta\to 0^+}\frac{\left(2\sin\frac{1}{2}\theta\,\cos\frac{1}{2}\theta\right)\cos^3\frac{1}{2}\theta}{\sin\frac{1}{2}\theta} = 2\lim_{\theta\to 0^+}\cos^4\left(\tfrac{1}{2}\theta\right) = 2(1)^4 = 2$$

21. (a) Distance = rate × time, so time = distance/rate. $T_1 = \dfrac{D}{c_1}$, $T_2 = \dfrac{2\,|PR|}{c_1} + \dfrac{|RS|}{c_2} = \dfrac{2h\sec\theta}{c_1} + \dfrac{D-2h\tan\theta}{c_2}$,

$T_3 = \dfrac{2\sqrt{h^2 + D^2/4}}{c_1} = \dfrac{\sqrt{4h^2+D^2}}{c_1}$.

(b) $\dfrac{dT_2}{d\theta} = \dfrac{2h}{c_1}\cdot\sec\theta\,\tan\theta - \dfrac{2h}{c_2}\sec^2\theta = 0$ when $2h\sec\theta\left(\dfrac{1}{c_1}\tan\theta - \dfrac{1}{c_2}\sec\theta\right) = 0 \quad \Rightarrow$

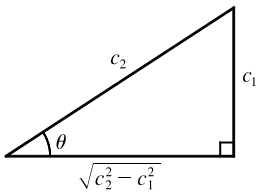
$\dfrac{1}{c_1}\dfrac{\sin\theta}{\cos\theta} - \dfrac{1}{c_2}\dfrac{1}{\cos\theta} = 0 \quad \Rightarrow \quad \dfrac{\sin\theta}{c_1\cos\theta} = \dfrac{1}{c_2\cos\theta} \quad \Rightarrow \quad \sin\theta = \dfrac{c_1}{c_2}$. The First Derivative Test shows that this gives a minimum.

(c) Using part (a) with $D = 1$ and $T_1 = 0.26$, we have $T_1 = \dfrac{D}{c_1} \quad \Rightarrow \quad c_1 = \frac{1}{0.26} \approx 3.85$ km/s. $T_3 = \dfrac{\sqrt{4h^2+D^2}}{c_1} \quad \Rightarrow$

$4h^2 + D^2 = T_3^2c_1^2 \quad \Rightarrow \quad h = \frac{1}{2}\sqrt{T_3^2c_1^2 - D^2} = \frac{1}{2}\sqrt{(0.34)^2(1/0.26)^2 - 1^2} \approx 0.42$ km. To find c_2, we use $\sin\theta = \dfrac{c_1}{c_2}$ from part (b) and $T_2 = \dfrac{2h\sec\theta}{c_1} + \dfrac{D-2h\tan\theta}{c_2}$ from part (a). From the figure,

$\sin\theta = \dfrac{c_1}{c_2} \quad \Rightarrow \quad \sec\theta = \dfrac{c_2}{\sqrt{c_2^2-c_1^2}}$ and $\tan\theta = \dfrac{c_1}{\sqrt{c_2^2-c_1^2}}$, so

$T_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \dfrac{D\sqrt{c_2^2-c_1^2} - 2hc_1}{c_2\sqrt{c_2^2-c_1^2}}$. Using the values for T_2 [given as 0.32],



h, c_1, and D, we can graph $Y_1 = T_2$ and $Y_2 = \dfrac{2hc_2}{c_1\sqrt{c_2^2-c_1^2}} + \dfrac{D\sqrt{c_2^2-c_1^2} - 2hc_1}{c_2\sqrt{c_2^2-c_1^2}}$ and find their intersection

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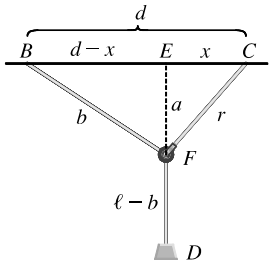
points. Doing so gives us $c_2 \approx 4.10$ and 7.66, but if $c_2 = 4.10$, then $\theta = \arcsin(c_1/c_2) \approx 69.6°$, which implies that point S is to the left of point R in the diagram. So $c_2 = 7.66$ km/s.

22. A straight line intersects the curve $y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2$ in four distinct points if and only if the graph of f has two inflection points. $f'(x) = 4x^3 + 3cx^2 + 24x - 5$ and $f''(x) = 12x^2 + 6cx + 24$.

$f''(x) = 0 \quad \Leftrightarrow \quad x = \dfrac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}$. There are two distinct roots for $f''(x) = 0$ (and hence two inflection points) if and only if the discriminant is positive; that is, $36c^2 - 1152 > 0 \quad \Leftrightarrow \quad c^2 > 32 \quad \Leftrightarrow \quad |c| > \sqrt{32}$. Thus, the desired values of c are $c < -4\sqrt{2}$ or $c > 4\sqrt{2}$.

23.



Let $a = |EF|$ and $b = |BF|$ as shown in the figure.

Since $\ell = |BF| + |FD|$, $|FD| = \ell - b$. Now

$$\begin{aligned} |ED| &= |EF| + |FD| = a + \ell - b \\ &\sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + a^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{(d-x)^2 + \left(\sqrt{r^2 - x^2}\right)^2} \\ &= \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 - 2dx + x^2 + r^2 - x^2} \end{aligned}$$

Let $f(x) = \sqrt{r^2 - x^2} + \ell - \sqrt{d^2 + r^2 - 2dx}$.

$f'(x) = \frac{1}{2}\left(r^2 - x^2\right)^{-1/2}(-2x) - \frac{1}{2}\left(d^2 + r^2 - 2dx\right)^{-1/2}(-2d) = \dfrac{-x}{\sqrt{r^2 - x^2}} + \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}}$.

$f'(x) = 0 \quad \Rightarrow \quad \dfrac{x}{\sqrt{r^2 - x^2}} = \dfrac{d}{\sqrt{d^2 + r^2 - 2dx}} \quad \Rightarrow \quad \dfrac{x^2}{r^2 - x^2} = \dfrac{d^2}{d^2 + r^2 - 2dx} \quad \Rightarrow$

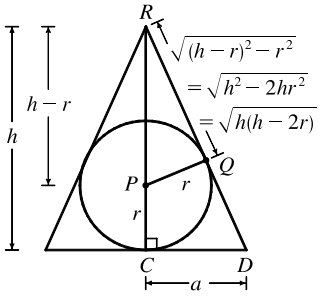
$d^2x^2 + r^2x^2 - 2dx^3 = d^2r^2 - d^2x^2 \quad \Rightarrow \quad 0 = 2dx^3 - 2d^2x^2 - r^2x^2 + d^2r^2 \quad \Rightarrow$

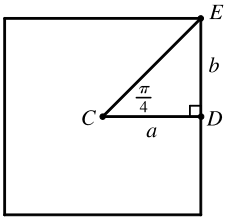
$0 = 2dx^2(x - d) - r^2\left(x^2 - d^2\right) \quad \Rightarrow \quad 0 = 2dx^2(x - d) - r^2(x + d)(x - d) \quad \Rightarrow \quad 0 = (x - d)\left[2dx^2 - r^2(x + d)\right]$

But $d > r > x$, so $x \neq d$. Thus, we solve $2dx^2 - r^2x - dr^2 = 0$ for x:

$x = \dfrac{-\left(-r^2\right) \pm \sqrt{\left(-r^2\right)^2 - 4(2d)(-dr^2)}}{2(2d)} = \dfrac{r^2 \pm \sqrt{r^4 + 8d^2r^2}}{4d}$. Because $\sqrt{r^4 + 8d^2r^2} > r^2$, the "negative" can be discarded. Thus, $x = \dfrac{r^2 + \sqrt{r^2}\sqrt{r^2 + 8d^2}}{4d} = \dfrac{r^2 + r\sqrt{r^2 + 8d^2}}{4d} \quad [r > 0] \quad = \dfrac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$. The maximum value of $|ED|$ occurs at this value of x.

24.





Let $a = \overline{CD}$ denote the distance from the center C of the base to the midpoint D of a side of the base.

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Since ΔPQR is similar to ΔDCR, $\dfrac{a}{h} = \dfrac{r}{\sqrt{h(h-2r)}} \quad \Rightarrow \quad a = \dfrac{rh}{\sqrt{h(h-2r)}} = r\dfrac{\sqrt{h}}{\sqrt{h-2r}}$.

Let b denote one-half the length of a side of the base. The area A of the base is

$A = 8(\text{area of } \Delta CDE) = 8\left(\frac{1}{2}ab\right) = 4a\left(a\tan\frac{\pi}{4}\right) = 4a^2$.

The volume of the pyramid is $V = \frac{1}{3}Ah = \frac{1}{3}\left(4a^2\right)h = \dfrac{4}{3}\left(r\dfrac{\sqrt{h}}{\sqrt{h-2r}}\right)^2 h = \frac{4}{3}r^2\dfrac{h^2}{h-2r}$, with domain $h > 2r$.

Now $\dfrac{dV}{dh} = \dfrac{4}{3}r^2 \cdot \dfrac{(h-2r)(2h) - h^2(1)}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h^2-4hr}{(h-2r)^2} = \dfrac{4}{3}r^2\dfrac{h(h-4r)}{(h-2r)^2}$

and

$$\begin{aligned}\frac{d^2V}{dh^2} &= \frac{4}{3}r^2 \cdot \frac{(h-2r)^2(2h-4r) - \left(h^2-4hr\right)(2)(h-2r)(1)}{[(h-2r)^2]^2} \\ &= \frac{4}{3}r^2 \cdot \frac{2(h-2r)[\left(h^2-4hr+4r^2\right) - \left(h^2-4hr\right)]}{(h-2r)^2} \\ &= \frac{8}{3}r^2 \cdot \frac{4r^2}{(h-2r)^3} = \frac{32}{3}r^4 \cdot \frac{1}{(h-2r)^3}.\end{aligned}$$

The first derivative is equal to zero for $h = 4r$ and the second derivative is positive for $h > 2r$, so the volume of the pyramid is minimized when $h = 4r$.

To extend our solution to a regular n-gon, we make the following changes:

(1) the number of sides of the base is n

(2) the number of triangles in the base is $2n$

(3) $\angle DCE = \dfrac{\pi}{n}$

(4) $b = a\tan\dfrac{\pi}{n}$

We then obtain the following results: $A = na^2\tan\dfrac{\pi}{n}$, $V = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h^2}{h-2r}$, $\dfrac{dV}{dh} = \dfrac{nr^2}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{h(h-4r)}{(h-2r)^2}$, and $\dfrac{d^2V}{dh^2} = \dfrac{8nr^4}{3}\cdot\tan\left(\dfrac{\pi}{n}\right)\cdot\dfrac{1}{(h-2r)^3}$. Notice that the answer, $h = 4r$, is independent of the number of sides of the base of the polygon!

25. $V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$. But $\dfrac{dV}{dt}$ is proportional to the surface area, so $\dfrac{dV}{dt} = k\cdot 4\pi r^2$ for some constant k.

Therefore, $4\pi r^2\dfrac{dr}{dt} = k\cdot 4\pi r^2 \quad \Leftrightarrow \quad \dfrac{dr}{dt} = k = \text{ constant}$. An antiderivative of k with respect to t is kt, so $r = kt + C$.

When $t = 0$, the radius r must equal the original radius r_0, so $C = r_0$, and $r = kt + r_0$. To find k we use the fact that when $t = 3$, $r = 3k + r_0$ and $V = \frac{1}{2}V_0 \quad \Rightarrow \quad \frac{4}{3}\pi(3k+r_0)^3 = \frac{1}{2}\cdot\frac{4}{3}\pi r_0^3 \quad \Rightarrow \quad (3k+r_0)^3 = \frac{1}{2}r_0^3 \quad \Rightarrow$

$3k + r_0 = \frac{1}{\sqrt[3]{2}}r_0 \quad \Rightarrow \quad k = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)$. Since $r = kt + r_0$, $r = \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0$. When the snowball

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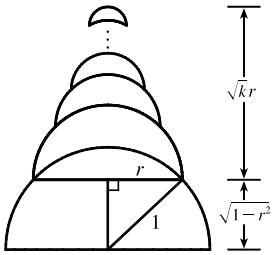
has melted completely we have $r = 0 \quad \Rightarrow \quad \frac{1}{3}r_0\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + r_0 = 0$ which gives $t = \dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1}$. Hence, it takes

$\dfrac{3\sqrt[3]{2}}{\sqrt[3]{2} - 1} - 3 = \dfrac{3}{\sqrt[3]{2} - 1} \approx 11$ h 33 min longer.

26. By ignoring the bottom hemisphere of the initial spherical bubble, we can rephrase the problem as follows: Prove that the maximum height of a stack of n hemispherical bubbles is $\sqrt{n}$ if the radius of the bottom hemisphere is 1. We proceed by induction. The case $n = 1$ is obvious since $\sqrt{1}$ is the height of the first hemisphere. Suppose the assertion is true for $n = k$ and let's suppose we have $k + 1$ hemispherical bubbles forming a stack of maximum height. Suppose the second hemisphere (counting from the bottom) has radius r. Then by our induction hypothesis (scaled to the setting of a bottom hemisphere of radius r), the height of the stack formed by the top k bubbles is $\sqrt{k}\,r$. (If it were shorter, then the total stack of $k + 1$ bubbles wouldn't have maximum height.)

The height of the whole stack is $H(r) = \sqrt{k}\,r + \sqrt{1 - r^2}$. (See the figure.) We want to choose r so as to maximize $H(r)$. Note that $0 < r < 1$.



We calculate $H'(r) = \sqrt{k} - \dfrac{r}{\sqrt{1 - r^2}}$ and $H''(r) = \dfrac{-1}{(1 - r^2)^{3/2}}$.

$H'(r) = 0 \quad \Leftrightarrow \quad r^2 = k(1 - r^2) \quad \Leftrightarrow \quad (k + 1)r^2 = k \quad \Leftrightarrow \quad r = \sqrt{\dfrac{k}{k + 1}}$.

This is the only critical number in $(0, 1)$ and it represents a local maximum (hence an absolute maximum) since $H''(r) < 0$ on $(0, 1)$. When $r = \sqrt{\dfrac{k}{k + 1}}$,

$H(r) = \sqrt{k}\dfrac{\sqrt{k}}{\sqrt{k + 1}} + \sqrt{1 - \dfrac{k}{k + 1}} = \dfrac{k}{\sqrt{k + 1}} + \dfrac{1}{\sqrt{k + 1}} = \sqrt{k + 1}$. Thus, the assertion is true for $n = k + 1$ when it is true for $n = k$. By induction, it is true for all positive integers n.

Note: In general, a maximally tall stack of n hemispherical bubbles consists of bubbles with radii

$$1, \sqrt{\frac{n - 1}{n}}, \sqrt{\frac{n - 2}{n}}, \ldots, \sqrt{\frac{2}{n}}, \sqrt{\frac{1}{n}}.$$

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